# Complex Analysis

## 1. Singularities at Infinity

For $f$, let $g(z) = f(1/z)$. If $g$ has a singularity at 0, then $f$ has a singularity at $\infty$.

## 2. The Residue Theorem

Let $G$ be a region (open and connected), $\partial G$ be piecewise $C^1$. Let $f \in Hol(Cl(G) - \{z_1, ..., z_n\})$, with $z_i \in G$. Then

$$\int_{\partial G} f(z)dz = 2\pi i \sum_{k=1}^{n} Res_{z_k} f$$

Note: $f$ holomorphic on a closed set $K$ means $\exists$ open $\Omega, K \subset \Omega, f \in Hol(\Omega)$

PROOF. Trivial for one singularity and simple regions of $G$ (such as triangles, ovals, etc.). In general, we triangularize the region $G$ as $G = \cup_j \Delta_j$, where $\delta_j$ contians no more than one singular point $z_k$. Then

$$\int_{\partial G} f(z)dz = \sum_j \int_{\Delta_j} f(z)dz = 2\pi i \sum_{k=1}^{n} Res_{z_k} f.$$

We should note that this proves assumes the existence of triangulation of a region $G$, a fundamental fact from topology. □

## 3. Defining log in $\mathbb{C}$

Let $z = |z|e^{i\theta}$. Then $Log(z) = \ln |z| + i(\theta + 2\pi k), k \in \mathbb{Z}$. Now let $\Omega$ be a simply connected and connected region, wiht $0 \notin \Omega$, and let $z_0 \in \Omega$. We now define

$$\log(z) := Log(z_0) + \int_{z_0}^{z} \frac{f'(\xi)}{f(\xi)}$$

## 4. The Argument Principle

Let $G$ be a bounded region with piecewise $C^1$ boundary, and let $z_k \in G$ for $k \in (1, ..., n)$, and $p_j \in G, j \in (1, ..., m)$. Let $f \in Hol(Cl(G) - \cup p_k)$, with poles at $p_k$, and zeroes at $z_k$ inside of $\partial G$. Then

$$\int_{\partial G} \frac{f'(z)}{f(z)} dz = 2\pi i (\text{number of zeroes - number of poles})$$

including multiplicity.

Proof. We have

$$f(z) = (z - z_k)^d g(z), \quad g(z_k) \neq 0.$$

Then we compute

$$\frac{f'(z)}{f(z)} = \frac{d(z - z_k)^{d-1} g(z)}{(z - z_k)^d g(z)} + \frac{(z - z_k)^d g'(z)}{(z - z_k)^d g(z)}.$$

The first term is a simple pole $\frac{d}{z - z_k}$, while the second is just a removable singularity. The proof follows from repeating this process, which obtains all zeroes with multiplicity, and integrating the entire expression. □