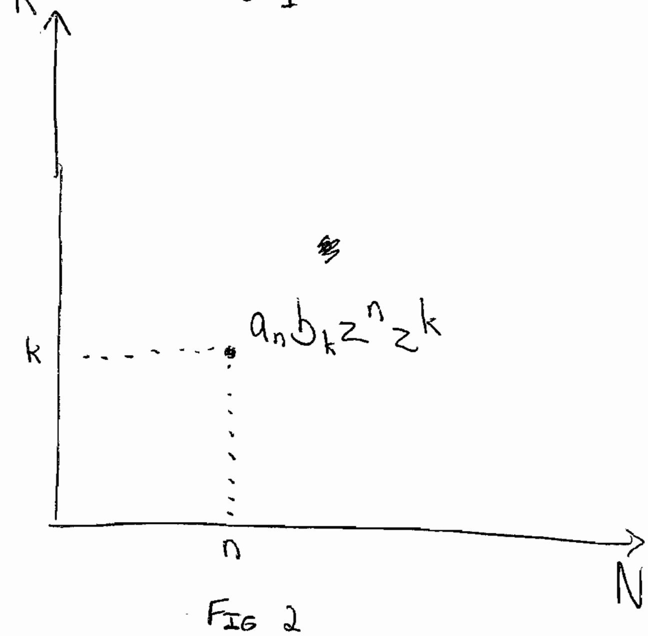
Proof Writing

ex: problem on product of power series

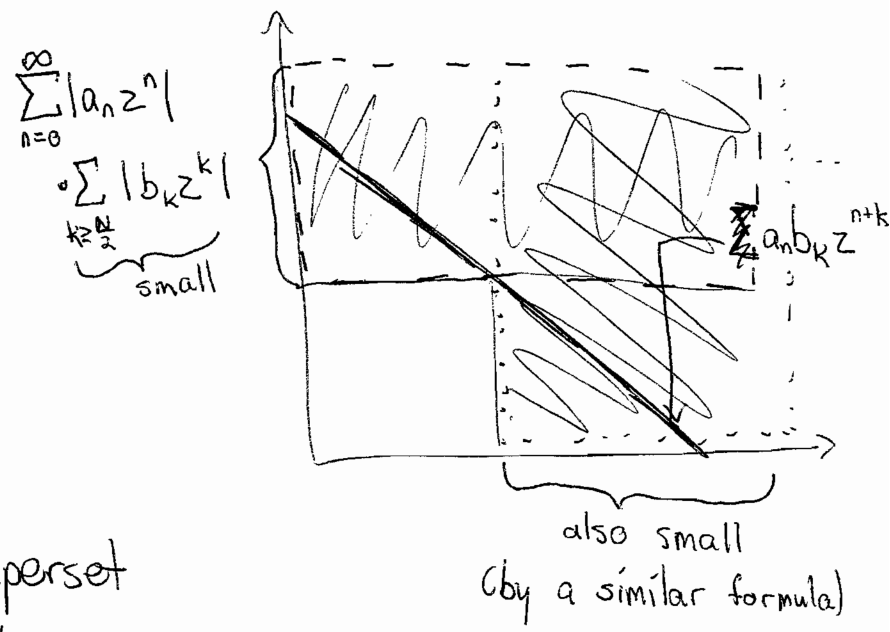
Thms: $(\sum a_n)(\sum b_i) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)$

FIG 1



We seek to show that everywhere above the line is small, so we show it by pieces.

FIG 2



As shown in FIG 2, a box strictly below the line can be found such that all points "above" or "to the right of" the box are small.

The union of these is a superset of the points above the line.

□

①

Rouche Thm.

Let G be a region with ∂G - PC'

Let $f, r \in Hol(Cl\ G)$

and assume $|f(z)| > |r(z)|$ $\forall z \in \partial G$ $\quad (\Rightarrow f(z) \neq 0 \ \forall z \in \partial G)$

Then the number of zeroes of f in G = the number of zeroes of $f+r$ in G.

Pf: $F(z) = \dfrac{f(z) + r(z)}{f(z)}$

It is sufficient to show that F has an equal number of poles and zeroes (in G).

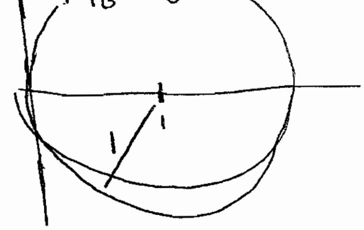
This would be shown by $\displaystyle\int_{\partial G} \frac{F'(z)}{F(z)}\, dz = 0$ (the argument principle)

$\left|1 - F(z)\right| = \left|1 - \frac{f+r}{f}\right| = \left|\frac{r}{f}\right| < 1$ on (some neighborhood of) ∂G

So $F(z) \in \{ w \in \mathbb{C} \mid |1-w| < 1 \}$

(see Fig 3)

Fig 3



$\log w$ is defined here;

$$\log w = 0 + \int_1^w \frac{dz}{z}$$

$\log F(z)$ is defined in a neighborhood of ∂G:

$$[\log F(z)]' = \frac{F'(z)}{F(z)}$$

So $\frac{F'(z)}{F(z)}$ has an anti-derivative in a neighborhood of ∂G

By , this must equal 0. $\square$

Corollaries:

1) Main thm. of algebra with localization of zeroes

$$z^n + \sum_{k=0}^{n-1} a_k z^k$$

2) Local structure of analytic functions

Let f be analytic in a neighborhood of z_0 and $f(z_0) = w_0$.

Let $n \geq 1$ be $\min\{k \geq 1 \mid f^{(k)}(z_0) \neq 0\}$

Then $\exists$ neighborhood W of w_0 (say, a disc with radius ε) and U of z_0

s.t. $f(z) = w$ has exactly n solutions in U $(\forall w \in W)$

Pf: We want to solve the equation, so write

$$f(z) = w$$

$$\Rightarrow f(z) - w = 0$$

$$\Rightarrow w_0 - w + a(z-z_0)^n + \underbrace{\sum_{k=n+1}^{\infty} a_k (z-z_0)^k}_{r(z)} = 0$$

$\exists\, \varepsilon > 0$ s.t. $\forall z$, $z - z_0 < \varepsilon \Rightarrow |r(z)| < \frac{1}{2}|a|\varepsilon^n$

If $|w - w_0| < \frac{1}{2}|a|\varepsilon^n$, then $|r(z)| < |w - w_0 + a(z-z_0)^n|$ (Triangle Inequality) if $|z - z_0| = \varepsilon$

So in the disc $|z - z_0| < \varepsilon$, and assuming $|w - w_0| < \frac{1}{2}|a|\varepsilon^n$, $f(z) = w$ has exactly as many solutions as

$$w_0 - w + a(z-z_0)^n = 0 \quad \text{(i.e. } n\text{)} \qquad \square$$

3) A holomorphic function is an open map

4) Maximum models principle:

if $f \in Hol(G)$ ~~and $f \not\equiv$ constant~~ and $f(z_0)$ is a local max.

then $f \equiv$ constant