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Conformal Maps

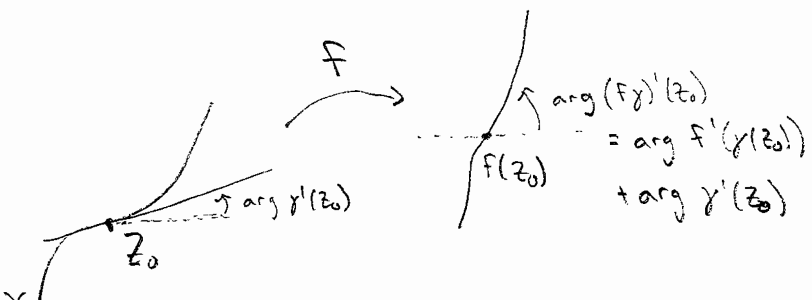
A map $f : \Omega \to G$, two domains in $\mathbb{C}$, is conformal if it is analytic & bijective.

f conformal $\Rightarrow f'(z) \neq \quad \forall z \in \Omega$

The converse is false: e^z is analytic and never has derivative zero, but it is not one-to-one

For $z_0 \in \Omega$, $f'(z_0) \neq 0 \Rightarrow f$ is locally conformal.
This Follows from the inverse function theorem

Example: Let $f = u + iv$. the Jacobian $J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix}$



$\det J = u_x^2 + u_y^2 = |f'(z)|^2$

$|f'(z_0)|$ describes how f locally stretches
and $\arg f'(z_0)$ describes how f locally rotates

Let $F : \mathbb{C} \to \mathbb{C}$ be conformal. At ∞, f has an isolated singularity.

Suppose $f(\infty) = c$, $c \neq \infty$, i.e. $\lim_{z \to \infty} f(z) = c$, $c \neq \infty$.

Then f is near c for large $|z|$, and by continuity f is bounded,
Thus f is constant, contradicting one-to-one.

If f is an essential singularity, then $f(\{z : |z| > 1\})$ is everywhere dense in $\mathbb{C}$.
now $f(\{z : |z| < 1\})$ is open by the open mapping theorem,
so $f(\{z : |z| < 1\}) \cap f(\{z : |z| > 1\}) \neq \emptyset$, contradicting one-to-one.

So f has a pole at ∞.

$f(z) = z^m g(z)$, $m > 0$ and $g(\infty) = c \neq \infty$,

$|f(z)| \leq C_0 z^m$ for large $|z|$, so f has m zeroes by Rouche's Theorem.

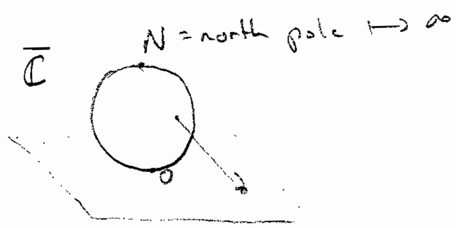
Thus $m = 1$, and f has the form $f = az + b$.

Let $\overline{\mathbb{C}}$ be the set $\mathbb{C} \cup \{\infty\}$ where $\{z : |z| > n\}$, $n \in \mathbb{N}$, is a neighbourhood base.

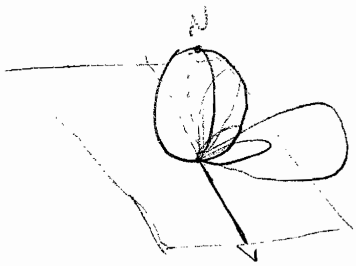
i.e., $\Omega \subseteq \overline{\mathbb{C}}$ is open if $\Omega \subseteq \mathbb{C}$ and Ω is open in $\mathbb{C}$

or $\Omega \setminus \{\infty\}$ is open in $\mathbb{C}$ and Ω contains a neighbourhood of ∞.

Stereographic Projection



Stereographic projection sends circles to circles. If the circle on the sphere passed through N, the its image is a line (here a generalized circle).



A conformal map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$:

$$f(\infty) = C \neq \infty, \quad g(z) = \frac{1}{f(z) - C} = az + b, \ a \neq 0$$

$$f(z) = \frac{1}{az+b} + C$$

$$f(z) = \frac{az+b}{cz+d} \quad \text{with} \quad ad - bc \neq 0.$$

Functions of the form $f(z) = \dfrac{az+b}{cz+d}$ form a group under composition,

and $\forall \underbrace{z_1, z_2, z_3}_{\text{distinct}}, \underbrace{w_1, w_2, w_3}_{\text{distinct}} \in \overline{\mathbb{C}}$, $\exists f(z) = \dfrac{az+b}{cz+d}$ s.t. $f(z_j) = w_j$, $j = 1, 2, 3$

Cross Ratio:

Consider $f(z) = \dfrac{z - z_1}{z - z_3} \cdot \dfrac{z_2 - z_3}{z_2 - z_1}$ So $f(z_1) = 0$, $f(z_2) = 1$, $f(z_3) = \infty$

define such a g for w_1, w_2, w_3. Then $g^{-1} \circ f$ sends z_j to w_j, $j = 1, 2, 3$

This is can be understood even if z_j or w_j is ∞.