Schwartz Lemma

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Sept 29/10

$f: \mathcal{D} \to \mathcal{D}$ analytic, $\mathcal{D} = \{|z| < 1\}$

Then $|f(z)| \leq |z|$, $z \in \mathcal{D}$

and if $|f(z_0)| = |z_0|$ for some $z_0 \in \mathcal{D}$,

then $f(z) = e^{i\alpha} z \quad \forall z \in \mathcal{D}$.

$|f'(0)| = 1$ also implies $f(z) = e^{i\alpha} z \ \forall z \in \mathcal{D}$

proof: define $g(z) = \begin{cases} \frac{f(z)}{z} & z \in \mathcal{D} \setminus \{0\} \\ f'(0) & z = 0 \end{cases}$

This is analytic since $\lim_{z \to 0} \frac{f(z)}{z} = f'(0)$,

so $\frac{f(z)}{z}$ has a removable discontinuity.

Let $z_0 \in \mathcal{D} \setminus \{0\}$. $|z_0| < r < 1$.

By maximum principle with g on $\{|z| \leq r\}$, $|g(z_0)| \leq \frac{1}{r}$

Letting $r \to 1$, we get that $|g(z)| \leq 1$ for $z \in \mathcal{D}$.

Thus $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

If $|f'(0)| = 1$ or $|f(z_0)| = |z_0|$, then $|g(z_0)| = 1$, a maximum of g.

Since g attains a maximum at an inside point, g is constant.

Maximum Principle.

$f \in Hol(\mathcal{D})$ and $f \in C(\bar{\mathcal{D}})$, and $|f(z)| \leq 1 \quad \forall z \in \partial\mathcal{D}$,

Then $|f(z)| \leq 1$ everywhere. This follows from the Cauchy integral formula that tells us in particular that an internal point takes the average value of the values in a circle around it.

Oliver Cheng Sept 27/10

$f : \mathcal{D} \to \mathcal{D}$ conformal

Consider $g_a(z) = \dfrac{z-a}{1-\bar{a}z}$, $a \in \mathcal{D}$

$$|g_a(z)|^2 = \frac{z-a}{1-\bar{a}z} \cdot \frac{\bar{z}-\bar{a}}{1-a\bar{z}}$$

$$= \frac{|z|^2 - \bar{a}z - a\bar{z} + |a|^2}{1 - a\bar{z} - \bar{a}z + |a|^2|z|^2}$$

We need only consider $|z| = 1$.

$= 1$. Since $|g_a(z)|^2 = 1$ on $\partial\mathcal{D}$,

$|g_a(z)|^2 < 1$ on $\mathcal{D}$ (strict inequality since $g_a \not\equiv c$.)

Note: $g_a(a) = 0$.

Since these maps form a group under composition,

$\forall\, a, b \in \mathcal{D}$, $g_b^{-1} \circ g_a$ sends $a \longmapsto b$.

Suppose $f(0) = a$, f conformal on $\mathcal{D}$, then $(g_a \circ f)(a) = 0$

By Schwartz, $|(g_a \circ f)'(0)| \le 1$

Since f and g are conformal, $(g_a \circ f)^{-1}$ is conformal,

so again $|((g_a \circ f)^{-1})'(0)| \le 1$. Thus $|(g_a \circ f)'(0)| \ge 1$.

Now Schwartz Lemma says $(g_a \circ f)(z) = e^{i\alpha} z$.

So $f(z) = g_a^{-1}(e^{i\alpha} z)$, some $a \in \mathcal{D}$, $\alpha \in \mathbb{R}$.

So $f : \mathcal{D} \to \mathcal{D}$ conformal has the form $f(z) = e^{i\alpha} \dfrac{z - \bar{a}}{1 - \bar{a}z}$ for some $a \in \mathcal{D}$, $\alpha \in \mathbb{R}$

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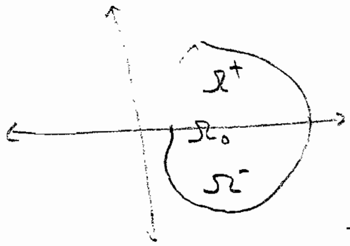
Symmetry Principle



$f \in H(\Omega^+)$

$f \in C(\Omega^+ \cup \Omega^0)$

f is real on Ω^0

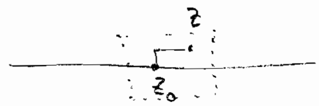
Then $\exists\, g \in H(\Omega)$ such that $g = f$ on $\Omega^+ \cup \Omega^0$.

Proof. Define
$$g(z) = \begin{cases} f(z) & z \in \Omega^+ \\ f(z) & z \in \Omega^0 \\ \overline{f(\bar{z})} & z \in \Omega^- \end{cases}$$

g is analytic on Ω^+ trivially,

On Ω^-, $g(z) = \overline{f(\bar{z})} = \sum \overline{a_n}\,(z - \bar{z}_0)^n$, so g is analytic.

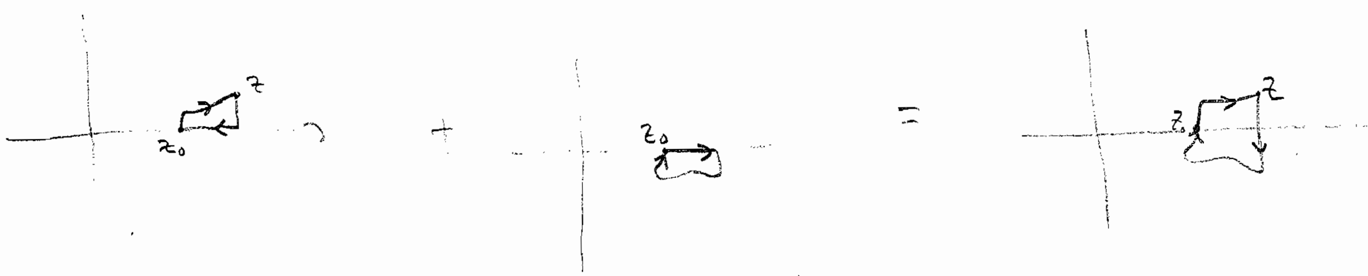
Let $z_0 \in \Omega^0$.



Take a small rectangle about z_0.

Let $F(z) = \int_\gamma f(z)\,dz$, γ a path from z_0 to z.

For two paths from z_0 to z, consider the closed loop they create



The upper and lower curves both have integral zero by continuity of the integral to include Ω^0. $\square$