- Conformal maps : $\mathbb{C} \to \mathbb{C}$ iff $f(z) = az + b$.
- Conformal maps : $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ iff LFT (linear fractional transformation) $f(z) = \dfrac{az+b}{cz+\omega}$
- Given S_1, S_2 LFT, so is $S_1 \circ S_2$, simply because S_1, S_2 analytic, bijective $\Rightarrow S_1 \circ S_2$ analytic bijective on $\hat{\mathbb{C}}$ $\Leftrightarrow$ LFT

  so LFT form a Group.

- Cross ratio.

  $\forall$ distinct $z_2, z_3, z_4 \in \hat{\mathbb{C}}$

  LFT S_{z_2, z_3, z_4} : $z_2 \longrightarrow 1$, $z_3 \longrightarrow 0$, $z_4 \longrightarrow \infty$

Case

I) $z_2, z_3, z_4 \in \mathbb{C} \Rightarrow S_{z_2 z_3 z_4}(z) = \dfrac{z - z_3}{z - z_4} \cdot \dfrac{z_2 - z_4}{z_2 - z_3}$

II) $z_2 = \infty$ $\quad S_{z_2, z_3, z_4}(z) = \dfrac{z - z_3}{z - z_4}$

III) $z_3 = \infty$ $\quad S_{z_2, z_3, z_4}(z) = \dfrac{z_2 - z_4}{z - z_4}$

IV) $z_4 = \infty$ $\quad S_{z_2, z_3, z_4}(z) = \dfrac{z - z_3}{z_2 - z_3}$

Def: $(z_1 : z_2 : z_3 : z_4) := S_{z_2, z_3, z_4}(z_1)$

Thm: Let T be LFT. Then $\forall z_1, z_2, z_3, z_4$ distinct in $\hat{\mathbb{C}}$

$$(Tz_1 : Tz_2 : Tz_3 : Tz_4) = (z_1 : z_2 : z_3 : z_4).$$

Pf Let $S := S_{z_2, z_3, z_4}$ $\qquad S : z_2 \to 1, \; z_3 \to 0, \; z_4 \to \infty$ $\qquad S \circ T^{-1} : Tz_2 \longrightarrow 1, \; Tz_3 \longrightarrow 0, \; Tz_4 \longrightarrow \infty$

$$(Tz_1 : Tz_2 : Tz_3 : Tz_4) = (S \circ T^{-1})(Tz_1)$$
$$= S(z_1)$$

Def: "Circle" (generalized circle) is either a circle or a straight line.

Theorem: If T is LFT, then

$$T(\text{"circle"}) = \text{"circle"}$$

pf: Sufficient to prove that:

$$S(z) = \frac{1}{z} \Rightarrow \begin{cases} S(\operatorname{Re} z = 1) \text{ is a circle.} \\ S(|z-1| = 1) = \text{line.} \end{cases}$$

there's a pole on this circle.

sufficient, as by some linear transformation, we can get a general LFT

Thm: $z, z_1, z_2, z_3 \in$ "Circle" $\Leftrightarrow$ $(z : z_1 : z_2 : z_3) \in \mathbb{R}$

Note: This theorem implies the previous theorem, as LFT preserves cross ratio

pf: $S(z) := S_{\substack{z_1, z_2, z_3 \\ \downarrow\ \downarrow\ \downarrow \\ 1\ \ 0\ \ \infty}}(z)$

$$\{ z \mid (z : z_1 : z_2 : z_3) \in \mathbb{R} \} = S^{-1}(\mathbb{R})$$

Let $z = S^{-1}(w)$, so $w = S(z)$

Let $S(z) = \dfrac{az+b}{cz+d}$

$$w \in \mathbb{R} \Leftrightarrow w = \bar{w}$$

$$\Leftrightarrow \frac{az+b}{cz+d} = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}$$

imaginary   sum is imaginary.   imaginary

$$\Leftrightarrow \quad (a\bar{c} - c\bar{a})|z|^2 + (a\bar{d} - c\bar{b})z + (b\bar{c} - d\bar{a})\bar{z} + (b\bar{d} - \bar{b}d) = 0$$

If $a\bar{c} - c\bar{a} = 0$, then it's a line

If $a\bar{c} - c\bar{a} \neq 0$, divide the equation by $a\bar{c} - c\bar{a}$, we get

$$|z|^2 + Az + \bar{A}\bar{z} + B = 0 \quad , \ B \in \mathbb{R}$$

then
$$|z + \bar{A}|^2 = (z + \bar{A})(\bar{z} + A)$$
$$= |z|^2 + zA + \bar{z}\bar{A} + |A|^2$$
$$= |A|^2 - B$$

so it is a non-empty circle (non-empty because it is the inverse image of the real line !)