Symmetry.

Def. z^* is called symmetric to z with respect to the circle ζ defined by z_1, z_2, z_3 if

$$(z^* : z_1 : z_2 : z_3) = \overline{(z : z_1 : z_2 : z_3)}$$

note $(\zeta : z_1 : z_2 : z_3) \in \mathbb{R} \Rightarrow \text{Im}(\ldots) = 0$

but how we can apply LFT's without altering this condition

- Let $z_1, z_2, z_3 \in \mathbb{R}$, the "circle" through $z_1, z_2, z_3 = \mathbb{R}$.
  $S = S_{z_1 z_2 z_3}$ used to define the cross ratio is an LFT with real coefficients. $\Rightarrow S(\overline{z}) = \overline{S(z)}$

- $z_1, z_2, z_3 \in \mathbb{T}$, $|z_k| = 1$ so $z_k \overline{z_k} = 1$, $z_k = \frac{1}{\overline{z_k}}$ then

$$\overline{(z : z_1 : z_2 : z_3)} = \left(\frac{1}{z} : \frac{1}{z_1} : \frac{1}{z_2} : \frac{1}{z_3}\right) = \left(\frac{1}{\overline{z}} : \frac{1}{\overline{z_1}} : \frac{1}{\overline{z_2}} : \frac{1}{\overline{z_3}}\right) = \left(\frac{1}{\overline{z}} : z_1 : z_2 : z\right.$$

$$z^* = \frac{1}{\overline{z}}$$

note that $|z||z^*| = R^2 = 1$

- LFT's preserve symmetry since they preserve the cross ratio

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__example__ $f(z) = \frac{z+1}{z-2}$ we want to find $f(\mathbb{D})$ with $\mathbb{D} = \{z : |z| < 1\}$

the center of $f(\mathbb{D})$ is symmetric to ∞

$f(2) = \infty$ so $f(2^*) =$ center of $f(\mathbb{D})$ and $2^* = \frac{1}{2}$

$$f\left(\tfrac{1}{2}\right) = \frac{\frac{1}{2}+1}{\frac{1}{2}-2} = \frac{3/2}{-3/2} = -1$$

now, $f(-1) = 0$ so we end up with a unit circle centered at -1

alternately, conformal transformations preserve angles.

the angle between the real line and the circle is $90°$ at 1 and -1. The coefficients are real so the real line is mapped to itself. This means $f(1)$ and $f(-1)$ must also form $90°$ angles with $\mathbb{R}$.

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general form of conformal automorphisms of $\mathbb{D}$

First, consider LFT's that map $\mathbb{D} \to \mathbb{D}$

let $\varphi(z_0) = 0$ then $\varphi(\frac{1}{\bar{z}_0}) = \infty$

so we have $\varphi(z) = c\frac{z - z_0}{1 - \bar{z}_0 z}$

we need $|\varphi(1)| = 1$ so $|c|\,|\frac{1 - z_0}{1 - \bar{z}_0}| = 1 \Rightarrow |c| = 1$

then $\varphi(z) = e^{i\theta}\frac{z - z_0}{1 - \bar{z}_0 z}$

Thm: if $\varphi : \mathbb{D} \to \mathbb{D}$ is a conformal map it is an LFT

pf: case 1 $\to \varphi(0) = 0$

we know that $|\varphi(z)| < 1 \ \forall z$ so by schwartz lemma

we have $|\varphi(z)| \leq |z| \Rightarrow |\varphi'(0)| \leq 1$

if $\psi = \varphi^{-1}$ $|\psi'(0)| \leq 1$ and $\varphi'(0) \cdot \psi'(0) = 1 \Rightarrow |\varphi'(0)| = 1 \Rightarrow \varphi(z) = e^{i\theta}$

case 2 $\to \varphi(z_0) = 0$

introduce $b_{z_0}(z) = \frac{z - z_0}{1 - \bar{z}_0 z}$

consider $\varphi \circ b_{z_0}^{-1}$