10/8/2010 Complex Analysis notes

Homology (singular)

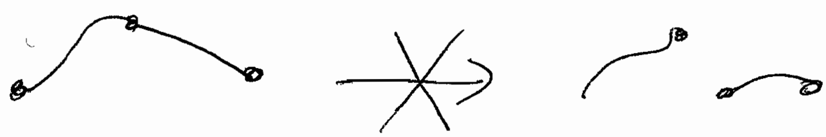
- $\forall$ 1-cycle, we can reparametrize all arcs

add oriented triangles - see

σ_1

extend to cont. map on whole region - everything cancels out except bdy of 2 chain

- Can add extra points to arcs
- Can transform cycles via homotopy (if we do not split the boundary)



- Can replace space by homotopy equivalent space

In particular we can replace a domain Ω by its homotopy deformation retract K

$K \subset \Omega$

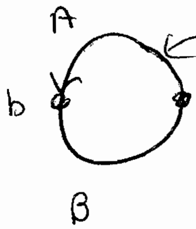
$\exists\ \varphi: \Omega \to K$ continuous $\varphi|_K = id$.

st $\varphi \sim id$.

Ex:

$H^1 \sim \mathbb{Z}$

$\swarrow$ retract

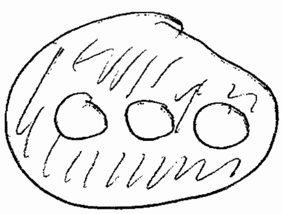


whenever we have this arc need ~~opposing~~ arc in B to compensate

$C = n_1 A + n_2 B \qquad \partial A = \{b\} - \{a\} \qquad \partial B = \{a\} - \{b\}$

$\partial C = 0 \Rightarrow n_1 = n_2$

Ex:



$\longrightarrow$

$H^1 \sim \mathbb{Z}^3$

these three cycles form a homology basis

Note: homotopy gp (π_1) is free gp. on three generators (noncommutative)

<u>Recognizing 0-cycles</u>

γ closed curve and $a \notin \gamma$

$$\mathrm{Ind}_a \gamma = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$$

<u>Claim:</u> $\mathrm{Ind}_a \gamma \in \mathbb{Z}$

<u>Pf</u> wlog γ has $\mathcal{C}^1$ parameterization

Take $u : \mathbb{T} \to \mathbb{R}$
$ v :$

$z : \mathbb{T} \to \mathbb{C}$

Use Lebesgue number lemma to replace pathological fcn. by piecewise affine fcns.
or convolve w/ a smooth kernel.

so $z = z(t)$ $\quad t \in [\alpha, \beta]$ $\quad z(\alpha) = z(\beta)$

Say $\alpha = 0$

$$h(t) = \int_\alpha^t \frac{z'(\tau)}{z(\tau)}\, d\tau$$

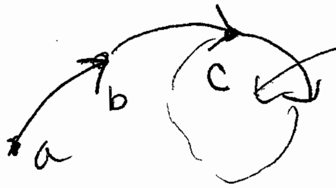
$$h'(t) = \frac{z'(t)}{z(t)} \Rightarrow \left(e^{-h(t)} z(t)\right)' = e^{-h(t)} z'(t) - e^{-h(t)} h'(t) z(t)$$
$$= 0$$

$$\Rightarrow \quad e^{-h(t)} z(t) \text{ is constant}$$

$$\frac{e^{h(t)}}{e^{h(0)}} = \frac{e^{h(t)}}{} = \frac{z(t)}{z(\alpha)} \Rightarrow e^{h(\beta)} = 1$$

$$\Rightarrow h(\beta) = 2\pi i k$$

γ-cycle $\Rightarrow$ $\gamma = \sum a_k \gamma_k$ $\quad a_k \in \mathbb{Z}$ $\quad \gamma_k$ - closed curves



delete this closed curve

repeat, deleting any closed curves along the way

Thm Let γ be a cycle in a region Ω

Then $\gamma \sim 0$ iff

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = 0 \quad \forall a \notin \Omega$$

$\Rightarrow$ trivial

$\Leftarrow$ ...

cont. fcns which take integer values are constant