COMPLEX ANALYSIS.

Eg: Find the residue of

$$\frac{1}{a^2+\sin^2 z}$$

If $f(z)$ has zero of order 1 at $z=z_0$, $\operatorname{res}_{z_0} \frac{1}{f} = \frac{1}{f'(z_0)}$

If $f(z)$ has zero of order 2 at $z=z_0$

$$\frac{1}{f(z)} = \sum_{k=-2}^{\infty} b_k (z-z_0)^k \Rightarrow \frac{(z-z_0)^2}{f} = \sum_{k=-2}^{\infty} b_k (z-z_0)^{k+2}.$$

$$\Rightarrow b_{-1} = \operatorname{res}_{z_0}\left(\frac{1}{f}\right) = \left[\frac{(z-z_0)^2}{f(z)}\right]'\Bigg|_{z=z_0}$$

If $f(z)$ has zero of order n then $\operatorname{Res}_{z_0} \frac{1}{f} = \left[\frac{(z-z_0)^n}{f(z)}\right]^{(n-1)}\Bigg|_{z=z_0}$

<u>Plemelj – Sokhotsky formula:</u>

Let γ be C^1 simple closed contour. Let $f \in C^1(\gamma)$

$\forall z \in \mathbb{C}\setminus\gamma$

$$F(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi \quad \text{is an analytic function.}$$

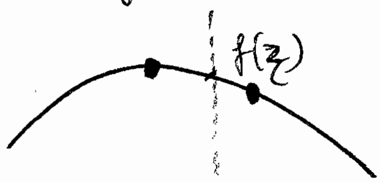
$$z_0 \in \gamma \quad \lim_{z\to z_0} F(z) = \frac{1}{2\pi i}\left(\text{p.v.}\int_\gamma \frac{f(\xi)}{\xi - z_0}\, d\xi\right) \pm \frac{1}{2} f(z_0)$$

$+$ if $z \to z_0$ from inside

$-$ if $z \to z_0$ from outside.

Proof: 1) Extend f to a C^1 function in a nbh of z_0.

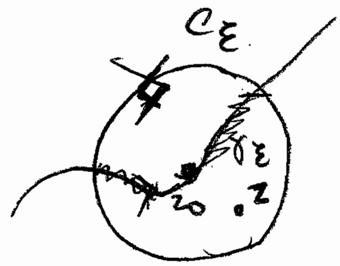
γ: $\begin{cases} y = \varphi(x). \\ \text{or } x = \psi(y). \end{cases}$ $\quad \varphi, \psi$ are C^1 functions.



$f(z) = f(\xi)$
where ξ is the unique point on γ
st $\operatorname{Re} z = \operatorname{Re} \xi$.

For any $\varepsilon > 0$

$$\int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi = \int_{\gamma_\varepsilon} + \int_{\gamma - \gamma_\varepsilon}$$



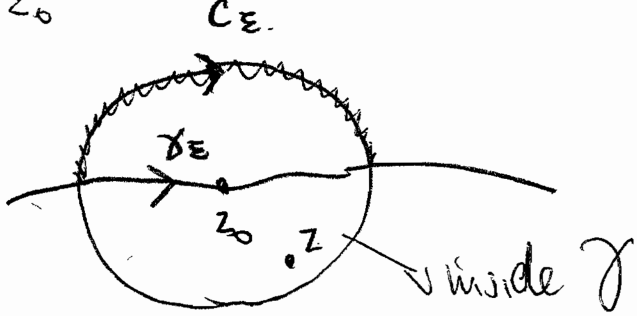
Let $\gamma_\varepsilon = \{ z \in \gamma : |z - z_0| < \varepsilon \}$

$$= \int_{\gamma \setminus \gamma_\varepsilon} \frac{f(\xi)}{\xi - z}\, d\xi + \underbrace{\int_{\gamma_\varepsilon} \frac{f(\xi) - f(z)}{\xi - z}\, d\xi}_{\leq} + \int_{\gamma_\varepsilon} \frac{f(z)}{\xi - z}\, d\xi$$

$$\downarrow z \to z_0 \qquad\qquad\qquad\qquad \downarrow z \to z_0$$

$$\int_{\gamma \setminus \gamma_\varepsilon} \frac{f(\xi)}{\xi - z_0}\, d\xi + M \cdot |\gamma_\varepsilon| + \int_{C_\varepsilon} \frac{f(z_0)}{\xi - z_0}\, d\xi$$

$$\int_{C_\varepsilon} \frac{f(z_0)}{\xi - z_0}\, d\xi = \pm \operatorname{Arc} C_\varepsilon \cdot f(z_0)$$



letting $\varepsilon \to 0$

$$\int_{\gamma \setminus \gamma_\varepsilon} \frac{f(\xi)}{\xi - z_0}\, d\xi \to p.v \int \frac{f(\xi)}{\xi - z_0}\, d\xi.$$

$$M\,|\gamma_\varepsilon| \to 0$$

$$\int_{C_\varepsilon} \frac{f(z_0)}{\xi - z_0}\, d\xi \to \pm \pi i\, f(z_0).$$