15 Oct.

Convergence of analytic functions. Normal families

$f_n \in Hol(\Omega)$

$f_n \to f$ always uniform convergence on compact sets unless otherwise specified

$\forall K \subset \Omega \quad \|f - f_n\|_{L^\infty(K)} \to 0$

this is preferred because

Lemma Let $f_n \in Hol(\Omega)$, $f_n \to f$ unif. on compacts.

Then:
1. $f \in Hol\,\Omega$
2. $f_n' \to f'$ (and so $f_n^{(k)} \to f^{(k)}$) as $n \to \infty$

Pf 1 follows from Morera theorem

Morera Thm: Let $f \in C(\Omega)$ s.t. $\int_\gamma f(z)dz = 0$ for "any" closed γ. Then f is analytic (i.e. $f \in Hol(\Omega)$)

Thm Let $f \in C(\Omega)$. TFAE:
1. $f \in Hol(\Omega)$
2. $\forall z_0 \in \Omega\ \exists r > 0$ s.t. $\int_{\partial R} f(z)dz$, $\forall$ rectangles $R \subset D(z_0, r) \subset \Omega$

Pf 2⇒1 is the interesting part.

2 ⇒ $\exists F$ s.t. $F_x = f$, $F_y = if$

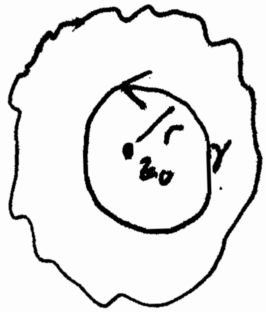
so $\bar\partial F = \frac{F_x + iF_y}{2} = 0 \Rightarrow F \in Hol(D(z_0, r))$

$f = F'$ so $f \in Hol(D(z_0, r))$ $\forall z_0$

so $R \subset \Omega$ $\int_{\partial R} f_n(z)dz = 0$

$\int_{\partial R} f(z)dz = \lim \int_{\partial R} f_n(z)dz$ by uniform convergence and compactness so $\int_{\partial R} f(z)\,dz = 0$, implying $f(z) \in Hol(\Omega)$

To prove 2: Observe $f_n \to f$ iff $\forall z_0 \in \Omega$ $\exists$ nbd U of z_0 s.t. $f_n \rightrightarrows f$ on U



$z : |z - z_0| \le r/2$

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^2}\,d\xi$$

$$f^{(n)}(z) =$$

$$|f'(z) - f_n'(z)| = \frac{1}{2\pi}\left|\int_\gamma \frac{f(\xi) - f_n(\xi)}{(\xi - z)^2}\right| \le \frac{1}{2\pi}\cdot 2\pi r \cdot \frac{\|f - f_n\|_{C(\gamma)}}{(r/2)^2}$$
$$= \frac{4}{r}\|f - f_n\|_{C(\gamma)} \to 0$$

so $f_n' \rightrightarrows f'$ on $D(z_0, r/2)$

Remark Let K_j - compact, $K_j \subset \Omega$ s.t. $\bigcup \operatorname{Int} K_j = \Omega$.
Then $f_n \to f$ unif on any compact $K \subset \Omega$ iff $f_n \rightrightarrows f$ on each K_j

construct $K_j = \{z \in \Omega : \operatorname{dist}(z, \Omega^c) \ge 2^{-j}\}$ if Ω bdd

Topology of $Hol(\Omega)$ defined by countably many $\|f\|_{C(K_j)}$

Prop Let $f, g \in Hol(\Omega)$ $d(f,g) = \sum_{j \ge 0} 2^{-j} \frac{\|f - g\|_{C(K_j)}}{1 + \|f - g\|_{C(K_j)}}$

Then: 1. d is a metric

2. $f_n \rightrightarrows f$ iff $d(f_n, f) \underset{n\to\infty}{\to} 0$

2 is trivial.

1: d is also translation-invariant (Frechet spaces = {Vector sp. topologized by translation invariant metric})

$d(f,f)=0$, $d(f,g)=d(g,f)$, $d(f,g)\geq 0$ all obvious

Lemma Let X,ρ be metric space. If $d(x,y)=\frac{\rho(x,y)}{1+\rho(x,y)}$,

$d(x,z)\leq d(x,y)+d(y,z)$ $\forall x,y,z\in X$.

Pf $t\to\frac{t}{1+t}$ = increasing function on $(0,\infty)$

so wlog $\rho(x,z)\geq\rho(x,y),\ \rho(y,z)$ (otherwise trivial)

$\rho(x,z)\leq\rho(x,y)$
$\Downarrow$
$\frac{\rho(x,z)}{1+\rho(x,z)}\leq\frac{\rho(x,y)}{1+\rho(x,y)}$

$$\frac{\rho(x,z)}{1+\rho(x,z)}\leq\frac{\rho(x,y)}{1+\rho(x,z)}+\frac{\rho(y,z)}{1+\rho(x,z)}$$

$$\leq\frac{\rho(x,y)}{1+\rho(x,y)}+\frac{\rho(y,z)}{1+\rho(y,z)}$$

so Δinequality holds for d