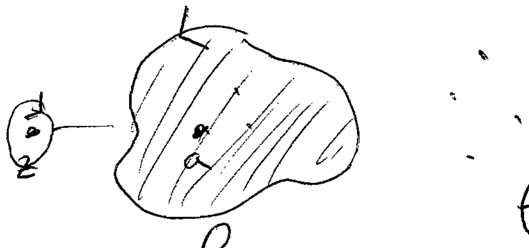
Complex analysis

From homework

on $\hat{\mathbb{C}}$ β:

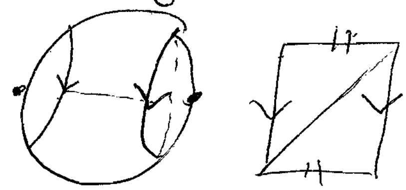


contours homologically equiv.

$\beta-$ poles inside

$\beta+$

on Riemann sphere equivalent



$$\beta = \beta_+ + \beta_- = \underbrace{\sum \frac{a_k}{(z-\lambda_k)^{n_k}}}$$

$$= \sum_{\lambda_k : \lambda_k \notin \Omega} + \sum_{\lambda_k : \lambda_k \in \Omega}$$

Consider $\displaystyle \frac{1}{2\pi i}\int_\gamma \frac{\beta(\zeta)}{\zeta - z} dz = \begin{cases} \beta_+(z) & z \in \Omega \\ -\beta_-(z) & z \notin \Omega \end{cases}$

because:

$\displaystyle \beta(z) = \frac{1}{(z-a)^k}$ $\quad a \notin \Omega \Rightarrow \frac{1}{2\pi i}\int_\gamma \frac{\beta(\zeta)}{\zeta - z} d\zeta = \begin{cases} \beta(z) & z \in \Omega \\ 0 & z \notin \Omega \end{cases}$

$\displaystyle a \in \Omega \Rightarrow \frac{1}{2\pi i}\int_\gamma \frac{\beta(\zeta)}{\zeta - z} d\zeta = \begin{cases} 0 & z \in \Omega \\ -\beta(z) & z \notin \Omega \end{cases}$

$\displaystyle \left(\text{use } \left|\int_{C_R}\right| \le 2\pi R \frac{1}{R}\frac{1}{R^k}\right)$

Say $\displaystyle \mathfrak{Z} = \frac{1}{\zeta - z} \Rightarrow \int_{\gamma^{-1}} \frac{\beta(\frac{1}{\zeta - \mathfrak{Z}})}{\cancel{\frac{1}{\zeta-\mathfrak{Z}} - z}} \left(\frac{-1}{(\zeta - \mathfrak{Z})^2}\right) d\mathfrak{Z}$

$\displaystyle = \cancel{\frac{1}{2\pi i}\int_{\gamma^{-1}} \beta}$

②

as $z \to \partial\Omega$ $\begin{cases} B_+(z) \\ -B_-(z) \end{cases} \longrightarrow \begin{aligned} &= \frac{1}{2\pi i} \, pv \int \cdots \;+\; \frac{1}{2}B(z) \\ &= \frac{1}{2\pi i} \, pv \int \;-\; \frac{1}{2}B(z) \end{aligned}$

adding: get $\frac{1}{2\pi i}\left(pv. \int + \cdots pv. \int \right)$

$B \in Hol(\mathbb{D})$

$\|B\|_\infty \leq 1$ $\quad B = \sum a_k z^k$ $\quad r<1$

$|a_k| \leq 1$ since $a_k = \frac{1}{2\pi i} \int_{|z|=r} \frac{B(z)}{z^{k+1}} \, dz$

$|a_k| \leq \frac{1}{r^k} \quad \forall r<1$

$B_n = \sum a_k(n) z^k$

$a_k(n) \xrightarrow[n\to\infty]{} a_k$

$\Rightarrow B_n \to B$

$\forall r<1$ show unif. conv. on $|z|<r$

$|B - B_n| \leq \left|\sum_{k=0}^{M} (a_k - a_k(n)) z^k\right| + \sum_{k=M+1}^{\infty} 2 \cdot r^k$

$\varepsilon > 0$ Pick M

st this sum

$< \varepsilon/2$

$$\lim \sum_{k=0}^{M} (a_k - a_k(n)) z^k = \sum_{k=0}^{M} \lim (a_k - a_k(n)) z^k = 0$$

$$\Rightarrow \sum_{k=0}^{M} (a_k - a_k(n)) z^k < \varepsilon/2 \quad \forall n > N$$

$\Rightarrow$ $f_n \in Hol(\mathbb{D})$

$\|f\|_\infty \le 1 \Rightarrow \exists n_k$ st $f_{n_k} \rightarrow f$

why? $f_n = \sum a_j(n) z^j$

$\exists$ subsequence n_k^0 of $1, 2, \ldots \ldots$

st $a_0(n_k^0) \underset{n_k^0 \to \infty}{\longrightarrow} a_0$

1	2	3	4
n_1^0	n_2^0	n_3^0	n_4^0
n_1^1	n_2^1	n_3^1	
n_2^1			

subseq of previous

$a_1(n_k^1) \underset{k \to \infty}{\longrightarrow} a_1$

$\Rightarrow a_2(n_k^2)$

Take diagonal

$\underline{a_j(n_{k+1}^k) \rightarrow a_j}$

Def A family $F \subset Hol(\Omega)$ is called locally bdd if $\forall z_0 \in \Omega$ $\exists$ nbd $U \ni z_0$ and $M < \infty$ st $\forall f \in F$ $|f(z)| \le M$ $\forall z \in U$.

Prop A family $F \subset Hol(\Omega)$ is relatively cmpt. if it is ~~l~~ locally bdd. (iff)

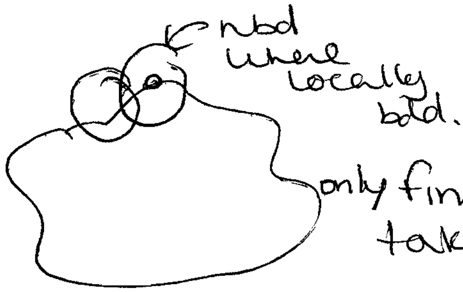
relatively cmpt = closure compact

We have proved if $F \subset Hol(\mathbb{D})$, $\|f\|_\infty \le 1$

no difference btn. 1 and M

$\mathbb{D}$ and arb. disc.

and



only finitely many nbds since cmpt
take subsequence of subsequence, etc.

"normal family" = "relatively cmpt family"

Comes from Arzela Ascoli