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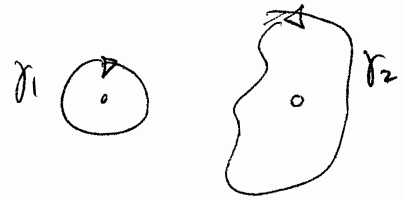
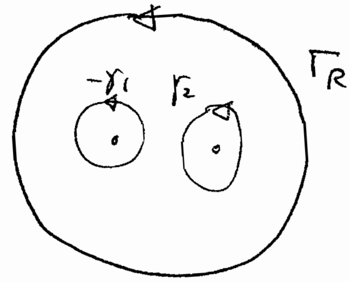
Why are γ_1 and γ_2 homologous?



Consider a circle Γ_R of radius large enough so that we get something like:



Notice that Γ_R is homologous to $-\gamma_1 + \gamma_2$. $\left(\int_{\Gamma_R} = \int_{-\gamma_1} + \int_{\gamma_2}\right)$

Since $\int_{\Gamma_R} \to 0$ as $R \to \infty$, we have $\int_{\gamma_1} = \int_{\gamma_2}$.

We want to prove the Riemann Mapping Theorem.

But first we need some technical theorems.

The following says that "analytic functions cannot appear from nowhere."

<u>Hurwitz' Theorem</u>/ Let $f_n(z) \in Hol(\Omega)$. If $f_n(z) \neq 0$ $\forall z \in \Omega$ and $f_n \to f$ uniformly on compact subsets. Then either $f \equiv 0$ or $f(z) \neq 0$ $\forall z \in \Omega$.

<u>Pf</u>/ Suppose $f \not\equiv 0$. If $f(z_0) = 0$ for some $z_0 \in \Omega$, then since zeros of analytic functions are isolated, $\exists r > 0$ s.t. $\overline{D(z_0; r)} \subset \Omega$ and $f(z) \neq 0$ $\forall z \in \overline{D(z_0; r)} - \{z_0\}$. By the argument principle,

$$0 \neq \begin{pmatrix} \text{mult. of } z_0 \\ \text{as a zero of } f \end{pmatrix} = \frac{1}{2\pi i} \int_{T_{z_0, r}} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \lim_{n \to \infty} \int_{T_{z_0, r}} \frac{f_n'(z)}{f_n(z)} dz = 0.$$

$f_n(z)$ is zero-free

This is a contradiction, so we conclude that $\nexists z_0 \in \Omega$ s.t. $f(z_0) = 0$.

$\Rightarrow\Leftarrow$

interchange of limit and integral is justified by uniform convergence on compact sets.

Riemann Mapping Theorem/ Let $\Omega \subset \hat{\mathbb{C}}$ (region) be "simply connected" and suppose that $\operatorname{card}(\hat{\mathbb{C}} - \Omega) > 1$ (i.e. $\hat{\mathbb{C}} - \Omega$ consists of more than one point). Let $z_0 \in \Omega$. Then $\exists!\ f: \Omega \to \mathbb{D}$ conformal s.t. $f(z_0) = 0$ and $f'(z_0) > 0$.

– Note that uniqueness is trivial. For if we have two such conformal maps $f_1, f_2 : \Omega \to \mathbb{D}$, then $\varphi = f_1 \circ f_2^{-1} : \mathbb{D} \to \mathbb{D}$ is conformal, has $\varphi(0) = 0$, and $\varphi'(0) > 0$, so that $\varphi(z) = z$.

– We put "simply connected" in quotes because there are two common definitions for simply connected commonly found in complex analysis texts. Our proof will show that they are equivalent.

- Closed paths are homotopic to points.
- Complements are connected

<u>Def</u>: $f \in \operatorname{Hol}(\Omega)$ is called <u>univalent</u> iff it is injective. (In other words, if f is a conformal map onto its image.)

<u>Pf</u>/ Let $\mathcal{F} = \{ g \in \operatorname{Hol}(\Omega) \text{ univalent} : \cancel{g(z_0) = 0}\ \ g'(z_0) > 0, \text{ AND } |g(z)| \le 1\ \forall z \in \Omega \}$.

Consider the problem of maximizing $g'(z_0)$ over all $g \in \mathcal{F}$. So let's try to show that $\exists f \in \mathcal{F}$ s.t. $f'(z_0) = \sup_{g \in \mathcal{F}} g'(z_0)$.

① <u>$\mathcal{F} \neq \emptyset$</u>. WLOG, we can assume that $0, \infty \notin \Omega$ by use of fractional linear transformations. Since Ω is simply connected (either definition), $\int_\gamma \frac{dz}{z} = 0$ for all closed curves $\gamma \subset \Omega$. Therefore, $\exists$(analytic) branch of $\log z$ on Ω, so $\exists$ branch of $g(z) = \sqrt{z} = \exp(\frac{1}{2} \log z)$.

Therefore, if $z \in g(\Omega)$, then $-z \notin g(\Omega)$.

Note that g is univalent, so $g: \Omega \to g(\Omega)$ is conformal.

$$\xleftarrow[z^2]{}$$

So $a \in g(\Omega) \Rightarrow D(a;r) \subset g(\Omega)$ for some $r > 0$

$$\Rightarrow D(-a;r) \cap g(\Omega) = \emptyset$$

Then $\left| \dfrac{1}{g(z)+a} \right| \le \dfrac{1}{r}$ in Ω.

Thus, we have constructed a bounded analytic function and it is univalent. If we multiply this function by an appropriate constant, we get a function in $\mathcal{F}$, so $\underline{\mathcal{F} \neq \emptyset}$.

To be continued.

Then $f(z) = \dfrac{r}{g(z)+a}$ is univalent in Ω

& satisfies $|f(z)| \le 1$

$f_1(z) = \dfrac{f(z) - f(z_0)}{1 - \overline{f(z_0)}\, f(z)}$ satisfies $f_1(z_0) = 0$

and $f_2(z) = \alpha f_1(z)$, $\alpha = \dfrac{|f_1'(z_0)|}{f_1'(z_0)}$

satisfies also $f_2'(z_0) > 0$.

So $f_2 \in \mathcal{F}$, so $\mathcal{F} \neq \emptyset$