Proof of Riemann Mapping Theorem cont'd

$f(z_0)=0$

$\mathcal{F} = \{ f : f$ is univalent in Ω s.t. $|f(z)| \leq 1 \ \forall z \in \Omega$ and $f'(z_0) > 0$ (real) $\}$

Proved that $\mathcal{F} \neq \emptyset$

Let $M = \sup_{g \in \mathcal{F}} g'(z_0)$, $M > 0$

Take $f_n \in \mathcal{F}$ s.t. $f_n'(z_0) \to M$

f_n is a bounded sequence. Thus, $\exists\, f_{n_K} \to f$ and $f_{n_K}' \to f'$.

Therefore, $f'(z_0) = M > 0$, and $f \neq$ constant

Show maximizer $f \in \mathcal{F}$

$|f_n(z_0)| \leq 1 \to |f(z)| \leq 1 \ \forall z$

Show f is univalent:

Fix $z_1 \in \Omega$. Then, $f_n(z) - f_n(z_1)$ are zero-free in $\Omega \setminus \{z_1\}$.

$f_n(z) - f_n(z_1) \longrightarrow f(z) - f(z_1)$

Either $f = $ const, or f has no zeroes in $\Omega \setminus \{z_1\}$ by Hurwitz.

Thus, $f(z) \neq f(z_1) \ \forall z \neq z_1$. Since z_1 is arbitrary, f is univalent.

Thus, $f \in \mathcal{F}$.

Claim: $f(\Omega) \supset \mathbb{D}$

Pf. Let $\exists\, a \in \mathbb{D}$ s.t. $a \notin f(\Omega)$

$$h_1(z) = \frac{f(z) - a}{1 - \bar{a} f(z)} \neq 0 \ \forall z \in \Omega$$

$h(z) = \sqrt{h_1(z)}$, allowed because h_1 is zero-free and Ω is simply connected

$$g(z) = \frac{h(z) - h(z_0)}{1 - \overline{h(z_0)}\, h(z)} \cdot \alpha, \quad |\alpha| = 1$$

$\hookrightarrow$ ensure $g'(z_0) > 0$

Claim $g \in \mathcal{F}$.

g is univalent because it is composition of univalent functions.

Claim $g'(z_0) > f'(z_0)$ (contradiction)

$h = \varphi_1(g)$, φ_1 is some LFT from $\mathbb{D} \to \mathbb{D}$, known as Möbius transformations (conformal automorphism of $\mathbb{D}$)

$h_1 = \varphi_2(h)$, $\varphi_2(z) = z^2$

$f = \varphi_3(h_1)$, φ_3 is Möbius

$\varphi = \varphi_3 \circ \varphi_2 \circ \varphi_1 : \mathbb{D} \to \mathbb{D}$

$\varphi(0) = 0$

φ is 2-1 (takes every value in its range twice)

φ is not az

$f = \varphi(g)$

$$f'(z_0) = \varphi'(\overset{0}{g(z_0)}) \cdot g'(z_0) = \varphi'(0) \cdot g'(z_0)$$

$|\varphi'(0)| < 1$ (by Schwartz Lemma)

$\varphi \in Hol(\mathbb{D})$ and $|\varphi(z)| \leq 1 \to \varphi'(0) < 1$ except when $\varphi(z) = az$, $a = $ constant

Thus, $g'(z_0) > f'(z_0)$

Therefore $\nexists\, a \in \mathbb{D}$ s.t. $a \notin f(\Omega)$ $\square$

Consider $\mathcal{F}' = \{ f \in Hol(\Omega)$

$|f(z)| \leq 1$

$f'(z_0) > 0 \}$

No longer require f to be univalent.

$\max_{g \in \mathcal{F}'} g'(z_0)$

By standard normal family argument, g exists.

Riemann mapping theorem does not make any statements about the boundary.

Harmonic Functions

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} = 4 \partial \bar{\partial}$$

$u \in C^2(\mathbb{R}^2)$ and $\Delta u = 0$

$\hookrightarrow$ NOT necessarily complex differentiable

If f is analytic, f is harmonic

$\rightarrow \bar{f}$ is harmonic

$\rightarrow \mathrm{Re}\, f = \frac{f + \bar{f}}{2}$ are harmonic

$\mathrm{Im}\, f = \frac{f - \bar{f}}{2i}$

Prop.: If u is harmonic and φ is analytic,

$\rightarrow u \circ \varphi$ is harmonic (assuming composition is defined)

$w = \varphi(z)$, $u = u(w) = u(\varphi(z))$

$$\frac{\partial u}{\partial z} = u_w \cdot w_z + \cancel{u_{\bar{w}} \cdot (\bar{w})_z}$$

$$\frac{\partial u}{\partial \bar{z}} = \cancel{u_w w_{\bar{z}}} + u_{\bar{w}} \cdot (\bar{w})_{\bar{z}}$$

$\varphi \in Hol$

$$\rightarrow \frac{\partial u}{\partial z} = u_w \varphi'(z)$$

$$\frac{\partial^2 u}{\partial \bar{z} \partial z} = \varphi'(z)\, u_{w\bar{w}} \overline{\varphi'(z)}$$

$$= \Delta_w u \cdot |\varphi'(z)|^2$$

$$\overset{\shortparallel}{0}$$

$$\frac{1 - |z|^2}{|1 - \bar{z}\zeta|^2}$$

$|z| < 1$

$|\zeta| \leq 1$

Δ_z, Δ_ζ

$\uparrow$

compute