Homework Problem:

$\dfrac{1-|z|^2}{|1-\bar{z}\xi|^2}$: If $f \in Hol(\mathbb{C})$, Then what is $\Delta |f|^2$?

$$\Delta |f|^2 = 4\partial\bar{\partial} f\bar{f} = 4\partial((\bar{\partial} f)\bar{f} + f\bar{\partial}\bar{f}), \text{ but } \bar{\partial}\bar{f} = \bar{f'}$$
$$= 4f'\bar{f'} = 4|f'|^2$$

So, $$\frac{1}{4}\Delta_\xi\left((1-|z|^2)\cdot\left|\frac{1}{1-\bar{z}\xi}\right|^2\right) = (1-|z|^2)\cdot\left|\frac{\partial}{\partial\xi}\left(\frac{1}{1-\bar{z}\xi}\right)\right|^2$$
$$= (1-|z|^2)\cdot\left|\frac{\bar{z}}{(1-\bar{z}\xi)^2}\right|^2$$

Also: $$\left|\frac{1}{1-\bar{\xi}z}\right|^2 - \left|\frac{z}{1-\bar{\xi}z}\right|^2 = \frac{1-|z|^2}{|1-\bar{z}\xi|^2}$$

Poissan Formula (Via Fourier Series):

Dirichlet Problem for $\mathbb{D}$: Given $f \in C(\mathbb{T})$, $\mathbb{T} = \{x \mid |x|=1\}$, Find a hamonic Function F in the $\mathbb{D}$, and continuous on closure$(\mathbb{D})$ such that $F|_{\mathbb{T}} = f$.

Brain Storm: Recall Fourier series, any function on the circle can be represented as $F(z) = \sum_{n\in\mathbb{Z}} \hat{F}(n) z^n$, $|z|=1$, $\hat{f}(n) = \int_{\mathbb{T}} F(z)\bar{z}^n \frac{dz}{2\pi}$

Rmk: If we take a parametrization $z = e^{it}$, for $\mathbb{T}$, Then $\hat{f}(n) = \int_0^{2\pi} F(e^{it}) e^{-int}\frac{dt}{2\pi}$

If $n \geq 0$, z^n is harmonic (in fact analytic), $n<0$ then z^n is not Harmonic. However $\bar{z}^{|n|}$ is Harmonic, and $\bar{z}^{|n|} = z^n$, for $|z|=1$.

If we set

Hence : $F(z) = \sum_{n \geq 0} \hat{F}(n) z^n + \sum_{n>0} \hat{F}(-n) \bar{z}^n$, for $|z| \leq 1$,

Then on the boundary we have our function f ! Using our definition for $\hat{F}$, we can write

$$F(z) = \sum_{n \geq 0} \left(\int_{\mathbb{T}} f(\zeta) \bar{\zeta}^n \frac{|d\zeta|}{2\pi} \right) z^n + \sum_{n>0} \left(\int_{\mathbb{T}} f(\zeta) \zeta^n \frac{|d\zeta|}{2\pi} \right) (\bar{z})^n$$

Interchanging $\int$ with $\sum$ we get :

$$F(z) = \int_{\mathbb{T}} P_z(\zeta) f(\zeta) \frac{|d\zeta|}{2\pi} \quad , \text{ where } P_z(\zeta) = \sum_{n \geq 0} z^n \bar{\zeta}^n + \sum_{n>0} \bar{z}^n \zeta^n$$

$$= \frac{1-|z|^2}{|1-\bar{z}|^2} = \frac{1-|z\zeta|^2}{|1-\bar{z}\zeta|^2}$$

for $|z|<1$, $|\zeta|=1$.

$P_z(\zeta)$ called Poissan Kernel (Didn't justify, just brainstorming)

Now,

Suppose we take arbitrary $f \in C(\mathbb{T})$, Define:

$$F(z) := \int_{\mathbb{T}} f(\zeta) P_z(\zeta) \frac{d\zeta}{2\pi}$$

Note: (1) F is harmonic in $\mathbb{D}$

(2) $\lim_{z \to z_0} F(z) = f(z_0)$, for $z_0 \in \mathbb{T}$

(1) Follows from the fact that sums are in $z^n, \bar{z}^n$, converge uniformly on compact subsets, and $z^n, \bar{z}^n, |\zeta|=1$, $\Rightarrow P_z(\zeta)$ is harmonic

(2) For this, use parametrization $z = re^{i\theta}$, $z_0 = e^{i\theta_0}$, $\zeta = e^{it}$

$$\Rightarrow F(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \frac{1-|r|^2}{|1-re^{i(\theta-t)}|^2} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i(\theta-t)}) \frac{1-r^2}{|1-re^{it}|^2} dt$$

Now, $\int \frac{1-r^2}{|1-re^{it}|^2} dt \cdot \frac{1}{2\pi} = \int P_r(\zeta) \frac{|d\zeta|}{2\pi} = 1$ — Power series representation.

Then $F(re^{i\theta}) - f(e^{i\theta_0}) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\left| f(e^{i(\theta-t)}) - f(e^{i\theta_0}) \right|}_{\downarrow 0} \cdot \frac{1-r^2}{|1-re^{it}|^2} dt$

as $r \to 1$

$\theta - t \to \theta_0$

$\forall \varepsilon > 0, \ \exists \delta$ s.t. $\forall s, t \in \mathbb{R}$ $\quad |s-t| < \delta \Rightarrow \quad |F(e^{is}) - F(e^{it})| < \frac{\varepsilon}{2}$

$\Uparrow$

<u>By uniform continuity of</u> f. Split integral into $\int_{-\delta/4}^{\delta/4} + \int_{Rest}$

Assume $|\theta - \theta_0| < \delta/4$. Then $\int_{-\delta/4}^{\delta/4} (\quad) < \varepsilon/2$, and $\int_{Rest} \to 0$ as $r \to 1$

Which gives the result. $\left(\frac{1-r^2}{|1-re^{it}|^2} \to 0 \text{ uniformly as } r \to 1 \right)$ $\boxtimes$

on $(-\pi, \pi) \setminus (-\delta, \delta)$
for all $\delta > 0$

So $\exists r_0 < 1$ s.t. $\forall r \in (r_0, 1)$

$$\int_{(-\pi,\pi)\setminus(-\delta,\delta)} |f(re^{i(\theta-t)}) \ f(e^{i\theta_0}))| \frac{1-r^2}{|1-re^{it}|^2} \frac{dt}{2\pi}$$

$$< \varepsilon/2$$

So $\forall \theta : |\theta - \theta_0| < \frac{\delta}{4}$

$\forall r \in (r_0, 1)$

$$|F(re^{i\theta}) - f(e^{i\theta_0})| < \varepsilon$$