Lecture # 22

Addition to proof of Riemann mapping theorem

$$\mathcal{F} := \left\{ \begin{array}{l} f \in Hol(\Omega), \text{ univalent in } \Omega \\ f'(z_0) > 0, \ |f(z)| \leq 1 \ \forall z \in \Omega \ \& \ \underline{f(z_0) = 0} \end{array} \right\}$$

For $\mathcal{F} \neq \emptyset$: • f univalent in Ω

$|f| \leq M$

• $f_1 = \frac{f}{M} \rightsquigarrow |f_1| \leq 1$

• $f_2 := \dfrac{f(z) - f(z_0)}{1 - \overline{f(z_0)} f(z)}$

• $f_3(z) = \alpha f_2(z)$ where $\alpha = \dfrac{|f_2'(z)|}{f_2'(z)}$

<u>Poisson - formula:</u>

$$F(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \bar{z}\xi|^2} f(\xi) \frac{|d\xi|}{2\pi}$$

with Poisson - Kernel $P_z = \dfrac{1 - |z|^2}{|1 - \bar{z}\xi|^2}$. Have convolution:

$$F(re^{i\theta}) = \int_{-\pi}^{\pi} P_r(e^{i(\theta - t)}) f(t) \frac{dt}{2\pi} = (P_r * f)(e^{it})$$

In general: $(f * g)(s) = \int_{\Gamma} f(t) g(s-t)\, dt = \int_{\Gamma} f(s-t) g(t)\, dt$ $\quad$ Γ: LCA-group with Haar-measure

E.g. $\Gamma =$ • $\mathbb{R}$

• $\mathbb{T}$ (as above)

• $\mathbb{Z}$ ($\int$ is Σ)

Convolution Theorem

$$\| \phi * f \|_{L^p} \leq \| \phi \|_{L^1} \cdot \| f \|_{L^p} \qquad 1 \leq p \leq \infty$$

no proof. $\square$

$\frac{\varepsilon}{3}$ - Theorem

Let $T, T_n : X \longrightarrow Y$ bounded, linear operators s.t.

$$\| T_n \| \leq M < \infty ,$$

X_0 dense in X & $T_n x \longrightarrow Tx \quad \forall x \in X_0$.

Then $T_n x \longrightarrow Tx \quad \forall x \in X$

no proof $\square$

Now consider

$$T_r f := P_r * f$$

Then $T_r f \xrightarrow{r \to 1} f$ $\forall$ trig. polynomial $f = \sum_{-N}^{N} a_k z^k$

because $T_r f = \sum_{-N}^{N} a_k r^{|k|} z^k$

Trig. polynomials are dense in $C(\mathbb{T})$ &

$$\| T_r f \|_{C(\mathbb{T})} = \| T_r f \|_\infty \leq \| f \|_{C(\mathbb{T})} = \| f \|_\infty$$

So T_r are uniformly bounded in $C(\mathbb{T})$ and $T_r f \underset{r \to 1}{\longrightarrow} f$ on a dense subset.

$\Longrightarrow$ By $\frac{\varepsilon}{3}$-Theorem : $\quad P_r * f \rightrightarrows f \quad \forall f \in C(\mathbb{T})$

Remark: $1 \leq p < \infty$

$$\| P_r * f - f \|_p \xrightarrow[r \to 1]{} 0 \qquad \forall f \in L^p(\mathbb{T})$$

Mean value properties

Thm: $u \in \mathrm{Harm}(\Omega)$, $D \Subset \Omega$, $D = D(z_0, r)$. Then,

$$u(z_0) = |\partial D|^{-1} \int_{\partial D} u(z)\, |dz| \qquad (MVP_1)$$

$$= \int_{-\pi}^{\pi} u(z + re^{it}) \frac{dt}{2\pi}$$

(MVP_1) implies

$$|D|^{-1} \int_D u(z)\, dx dy = u(z_0) \qquad (MVP_2)$$

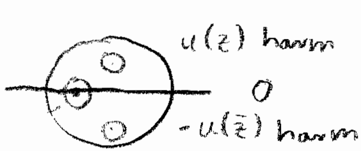
Def.: • Let $u \in C(\Omega)$. We say that $u \in (wMVP_1)$, if $\forall z_0 \in \Omega$ $\exists r_0 > 0$ s.t.

$$(wMVP_1) \qquad u(z_0) = \int_{-\pi}^{\pi} u(z_0 + re^{it}) \frac{dt}{2\pi} \qquad \forall r \in (0, r_0)$$

• $u \in (wMVP_2)$, if $\forall z_0 \in \Omega$ $\exists r_0 > 0$ s.t. $\forall r \in (0, r_0)$

$$u(z_0) = |D(z_0, r)|^{-1} \int_{D(z_0, r)} u(z) \underbrace{dA(z)}_{dx dy} \qquad (wMVP_2)$$

Goal: $u \in (wMVP_2) \Rightarrow u \in \mathrm{Harm}(\Omega)$

E.g.: 

on line (wMVP) holds because of symmetry

Thm.: (Maximum principle)

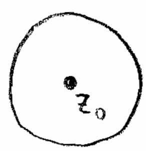
Let $u \in C(\Omega)$, Ω region, $u \in (wMVP_2)$

u real-valued

and let $z_0 \in \Omega$.

If u has local maximum or minimum at z_0, then $u \equiv \text{const}$.

Pf.: $\exists\, D(z_0, r)$ s.t. $u(z_0) = |D(z_0,r)|^{-1} \int_{D(z_0,r)} u(z)\, dA(z)$



If $u(z_0) = \max_{z \in D(z_0,r)} u(z)$, then

- $u(z) = u(z_0) \quad \forall z \in D$

or

- $\exists\, z \in D$ s.t. $u(z) < u(z_0)$

$$\Rightarrow \int_{D(z_0,r)} u(z)\, dA(z) < |D(z_0,r)|\, u(z_0)$$