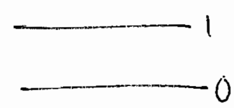
10/29/10

HW Review

Harmonic functions in the strip $0 \le \operatorname{Im} z \le 1$ with $u(x,0)=0$, $u(x,1)=1$.



$$u(z) = \operatorname{Im} z \quad \text{works.}$$

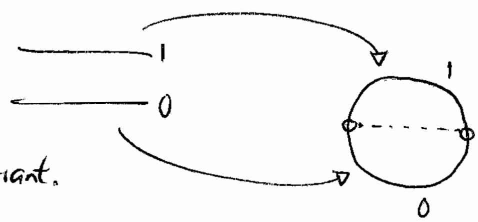
Is it unique? No, because we can always add the function

$$\operatorname{Im} e^{2\pi z}.$$

Note that when $\operatorname{Im} z = 0$, $\operatorname{Im} z = 1$, we have $z = x$, $z = x+i$, and $e^{(x+i)2\pi} = e^{2\pi x}$.

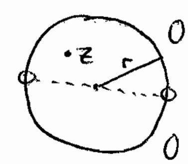
OK, but what if we also assume that the function is bounded? Is it unique then? Yes.

Use conformal mapping:



* Harmonicity is a conformal invariant.

Well, suppose there exists another bounded harmonic function with the same boundary values. Then their difference has to be another bounded harmonic function — taking the value 0 on the circle (possibly minus two points). We claim that this function is zero on the interior. To see this, use the Poisson Formula:



$$v(z) = \int_{\mathbb{T}} \underbrace{\frac{1-|z/r|^2}{|1-\bar{z}/r \cdot \xi|^2}}_{\substack{\downarrow \\ \text{Poisson kernel} \\ \text{as } r\to 1^-}} \underbrace{v(r\xi)}_{\substack{\downarrow \\ 0 \\ \text{as } r\to 1^-}} \frac{|d\xi|}{2\pi} \to 0 \text{ a.e. as } r \to 1^- \quad \text{by Dominated Convergence Theorem.}$$

In fact, if you look at this proof, you can see that we can further reduce the boundedness requirement. (Maybe L^1 with unif bdd norm?)

Recall that our goal is to prove:

$$wMVP_2 \Rightarrow \text{Harmonic}$$

To this end, we need:

Thm (Maximum Principle)/ Let $u \in C(\Omega)$, $u \in wMVP_2$. If u has a ~~local~~ global max [redacted] (or min) at $z_0 \in \Omega$, then $u \equiv$ constant.

Pf/ Suppose $u(z_0) \geq u(z)$ $\forall z \in D = D(z_0; r)$, and

$$u(z_0) = \frac{1}{|D|} \int_D u(z)\, dA(z).$$

Let $\exists\, z_1$ with $|z_0 - z_1| < r$ and $u(z_1) < u(z_0)$. (Assume this for the sake of contradiction.)

Then $\exists\, \varepsilon, \delta > 0$ s.t. $D' = D(z_1; \delta) \subset D$ and $\forall z \in D'$: $u(z) < u(z_0) - \varepsilon$. (This follows from continuity.)

Then [redacted] $u(z_0) = \frac{1}{|D|} \int_D u(z)\, dA(z) \leq u(z_0) - \frac{\varepsilon |D'|}{|D|} < u(z_0)$.

$\Rightarrow\Leftarrow$

Thus, $u(z)$ is constant in D.

Consider $\underbrace{\{z \in \Omega : u(z) = u(z_0)\}}_{K}$. This set is closed by virtue of being the preimage of a closed set. Also, $\forall z_1 \in K$, z_1 is a local max, so $\exists\, D(z; r)$ s.t. $u(z) = u(z_1) = u(z_0)$, so K is open. Thus, $K = \Omega$ (by "continuous induction".)

Prop: Let $u \in wMVP_2(\Omega)$. Then $u \in Harm(\Omega)$.

Pf/ Take an arbitrary $z_0 \in \Omega$. Take $r > 0$ s.t. $D = D(z_0; r) \Subset \Omega$. Let v denote the Poisson extension of $u|_{\partial D}$. Then $v \in wMVP_2(D)$, so $u - v \in wMVP_2(D)$.

Let $M = \max\{u(z)-v(z) : z \in \mathrm{Cl}\, D\}$

$m = \min\{u(z)-v(z) : z \in \mathrm{Cl}\, D\}$.

We claim that $M = 0$ — because if $M>0$, then $M = u(z_0)-v(z_0)$, $z_0 \in D$,

so $u-v$ has a global max at z_0,

so $u-v = \text{constant} = 0$

by continuity.

Similarly with the minimum.

$\therefore$ $u = v$, so in particular, u is harmonic.

Since we know the Poisson formula for the disc, we can get a Poisson formula for the upper half plane, the infinite strip, etc., by conformal mapping.