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Harmonic Conjugate

u - real valued harmonic function.

For v s.t. $u + iv$ is analytic, then v is called a harmonic conjugate.

① Let $u(z) = \sum_{k\geq 0} a_k z^k + \sum_{k>0} \bar{a}_k \bar{z}^k$, $a_0 \in \mathbb{R}$.

$= \sum_{k\geq 0} a_k z^k + \sum_{k>0} a_{-k} \bar{z}^k$ where $a_{-k} = \bar{a}_k$.

Choose $v = i \sum_{k>0} a_{-k} \bar{z}^k - i \sum_{k>0} a_k z^k + c$, $c \in \mathbb{R}$. $\quad ia_{-k} = \overline{-ia_k}$

② Consider $\dfrac{1-|z|^2}{|1-\bar{z}\zeta|^2} = \mathrm{Re}\left(\dfrac{\zeta+z}{\zeta-z}\right)$.

If $u \in \mathrm{Harm}(\mathbb{D}) \cap C(\mathrm{Cl}\,\mathbb{D})$, define $f(z) = \int_{\mathbb{T}} \dfrac{\zeta+z}{\zeta-z} u(\zeta) \dfrac{|d\zeta|}{2\pi}$.

Then $f(z)$ is analytic in $\mathbb{D}$ and $\mathrm{Re}\, f(z) = u(z)$. To find $v(z)$, take $\mathrm{Im}\, f(z)$.

↳ ($\because$ u is cont. $\rightarrow \int$ is bounded & $\dfrac{\zeta+z}{\zeta-z}$ is analytic).

$\dfrac{\zeta+z}{\zeta-z}$ is called Schwartz kernel.

How to compute complex conjugate?

$u + iv \in \mathrm{Hol}$

$\Downarrow$

$\dfrac{\partial}{\partial \bar{z}}(u+iv) = 0$. or $\dfrac{\partial}{\partial \bar{z}} u = -i \dfrac{\partial}{\partial \bar{z}} v$, $\dfrac{\partial}{\partial \bar{z}} v = i \dfrac{\partial}{\partial \bar{z}} u$.

then solve this equation.

$$d\omega = -u_{uu}\, dy \wedge dx + u_{xx}\, dx\, dy = \Delta u\, dx\, dy = 0$$

① In (x,y), we have $v_x + iv_y = i(u_x + iu_y)$ or $\begin{cases} v_x = -u_y \\ v_y = u_x \end{cases}$

$dv = \underbrace{-u_y\, dx + u_x\, dy}_{\omega}$ If $d\omega = 0 \Rightarrow$ closed $\Rightarrow$ exact in simply connected domain.

So, to get v, calculate $v(z) = \int_{z_0}^{z} \omega$.

② In z, $\bar\partial v = i\bar\partial u$

$$\left(\frac{\partial}{\partial \bar z} v\right) d\bar z = \underbrace{i \frac{\partial u}{\partial \bar z} d\bar z}_{\omega}.$$

If $d\omega = 0 \Rightarrow$ closed.

$dF = \partial F + \bar\partial F = F_z\, dz + F_{\bar z}\, d\bar z.$

$$d\omega = \cancel{i \frac{\partial^2 u}{\partial \bar z^2} d\bar z \wedge d\bar z} + i \frac{\partial^2 u}{\partial z \partial \bar z} dz \wedge d\bar z = \frac{i}{4} \underbrace{\Delta u}_{=0} dz \wedge d\bar z.$$

$$\left.\begin{aligned} \frac{\partial}{\partial \bar z} u &= F \\ \bar\partial u &= w \end{aligned}\right\}$$ d-bar equation. condition for local stability is $dw = 0$.

u, v - real so $\frac{\partial u}{\partial \bar z} = \overline{\left(\frac{\partial u}{\partial z}\right)} =$

so $\frac{\partial v}{\partial z} = -i \frac{\partial u}{\partial z}$

$$\left(\frac{\partial v}{\partial \bar z}\right) d\bar z + \left(\frac{\partial v}{\partial z}\right) dz = \underbrace{i\left(\frac{\partial u}{\partial \bar z}\right) d\bar z - i\left(\frac{\partial u}{\partial z}\right) dz}_{\omega}$$

This equation has unique solution (up to a constant) in a simply connected domain

$d\omega = 0$ because u is harmonic