Today 11/5

Notes by Will Berkeley

Two topics:

(1) Harnak's Inequality

Thm Let $u \geq 0$ harmonic in $|z - z_0| < R$. Then $\forall z$ s.t. $|z - z_0| \leq r$, $r < R$,

$$\frac{R-r}{R+r} u(z_0) \leq u(z) \leq \frac{R+r}{R-r} u(z_0)$$

Pf WLOG $z_0 = 0$, $R = 1$ by translation and rescaling

For the moment, assume $u \in C(\mathrm{cl}\, \mathbb{D})$ (u continuous on closed unit disc)

Then $$u(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{z}\xi|^2} u(\xi) \frac{|d\xi|}{2\pi}$$ (Poisson integral formula)

so $$u(z) \leq \frac{1-|z|^2}{(1-|z|)^2} \underbrace{\int_{\mathbb{T}} u(\xi) \frac{|d\xi|}{2\pi}}_{u(0)} = \frac{1+|z|}{1-|z|} u(0) \leq \frac{1+r}{1-r} u(0)$$

Holds because $\frac{1+x}{1-x}$ is inc. fn.

and $$u(z) \geq \frac{1-|z|^2}{(1+|z|)^2} u(0) = \frac{1-|z|}{1+|z|} u(0) \geq \frac{1-r}{1+r} u(0)$$

Now eliminate assumption $u \in C(\mathrm{cl}\, \mathbb{D})$. Consider

$$u(\rho z) \quad \rho < 1$$

$u(\rho z)$ is Harmonic in $D_{0, 1/\rho}$

$$\frac{1-|z|}{1+|z|} u(0) \leq u(\rho z) \leq \frac{1+|z|}{1-|z|} u(0)$$

Let $\rho \to 1^-$ to obtain

$$\frac{1+|z|}{1-|z|} u(0) \leq u(z) \leq \frac{1+|z|}{1-|z|} u(0)$$ and hence $$\frac{1-r}{1+r} u(0) \leq u(z) \leq \frac{1+r}{1-r} u(0)$$

QED

② Subharmonic functions

Want solve general Dirichlet problem: domain Ω, $u \in C(\partial\Omega)$

Find $h \in \mathrm{Harm}(\Omega) \cap C(\mathrm{cl}(\Omega))$ s.t. $h|_{\partial\Omega} = u$

A solution exists for almost all domains e.g. for all simply connected domains.

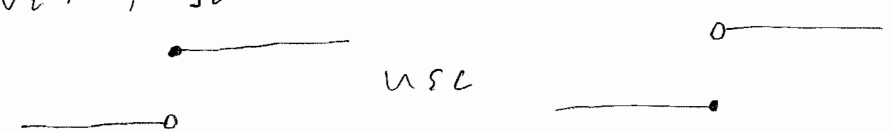
We approach the problem by introducing a wider class of functions.

<u>Def</u> $u : \Omega \to [-\infty, \infty]$ is <u>upper semicontinuous</u> (usc) if for any $z_0 \in \Omega$,

$$\limsup_{z \to z_0} u(z) \le u(z_0)$$

Compare with definition of continuity:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. } |z - z_0| < \delta \Rightarrow u(z_0) - \varepsilon < \underbrace{u(z) < u(z_0) + \varepsilon}_{\text{upper semicontinuous}}$$



<u>Rem</u> If u_n cont., $u_n \searrow u \Rightarrow u$ is usc

<u>Thm</u> Let K be compact Hausdorff and let $u \in USC(K)$ (i.e. u is upper semicontinuous on K). Then $\exists u_n \in C(K)$ s.t.

$$u_{n+1} \le u_n, \quad \lim_{n \to \infty} u_n = u$$

So any usc fn. on a compact Hausdorff space is the limit of a decreasing sequence of continuous fn.s

<u>pf</u> Omitted. Will apply it only to compact subsets of $\mathbb{C}$.

<u>Def</u> A fn. $u : \Omega \to [-\infty, \infty]$ is called subharmonic if

1. u is usc
2. w. MVI. holds, i.e. $\forall z_0 \in \Omega\ \exists r_0$ s.t. $u(z_0) \le \int_{|z - z_0| = r} u(z) \frac{|dz|}{2\pi} \quad \forall r < r_0$

We can integrate because usc fn. is the decreasing limit of continuous functions on a compact Hausdorff space, so is measurable.

Ex $f \in Hol(\Omega)$

then $\log|f| \in SH(\Omega)$ (i.e. is subharmonic in Ω)

If $f(z_0) \neq 0$, $\log f(z)$ is defined in a nbh of z_0, so $\log|f(z)| = \mathrm{Re}\log f(z)$ is harmonic (in the nbh of z_0)

If $f(z_0) = 0$, $\log|f(z_0)| = -\infty$, so the required inequality is trivially satisfied.

Rmk $wMVI_1 \Rightarrow wMVI_2$, meaning

$\forall z_0, \exists r_0$ s.t. $$u(z_0) \le \frac{1}{\pi r^2}\int_{D_{z_0,r}} u(z)\, dA(z)$$

Why subharmonic? Because dominated by harmonic fns in a way we will now describe.

Lemma Let $u \in SH(\Omega)$ and suppose u has a maximum at $z_0 \in \Omega$. Then $u = $ constant.

Pf essentially the same proof as for harmonic functions using $wMVI_1$ and continuous induction

Thm (strong max principle) Let Ω be a bounded region. Let $u \in SH(\Omega)$ s.t. $\forall \xi \in \partial\Omega$

$$\limsup_{z \to \xi} u(z) \le 0$$

Then $u(z) \le 0 \ \forall z \in \Omega$

Pf Suppose $M = \sup_{z \in \Omega} u(z) > 0$. Derive a contradiction

Take z_n s.t. $u(z_n) \to M$. By compactness of $Cl(\Omega)$,

$\exists z_{n_k} \to z_0 \in Cl(\Omega)$

$$\limsup_{z \to z_0} u(z) \geq \lim u(z_{n_k}) = M > 0$$

we conclude that $z_0 \notin \partial\Omega$, so $z_0 \in \Omega$ and, by usc,

$u(z_0) \geq \limsup u(z) \geq M$.

But M is the supremum, so $u(z_0) = M$ and u has a maximum in Ω. Thus $u(z) \equiv M$ in Ω

$\Rightarrow\Leftarrow$ $\limsup_{z \to \beta} u(z) \leq 0$

QED

<u>Cor</u> Let $u \in SH(\Omega)$ and $v \in \mathrm{Harm}(\Omega)$ s.t.

$$\limsup_{z \to \partial\Omega} (u(z) - v(z)) \leq 0.$$

Then $u(z) \leq v(z)$ $\forall z \in \Omega$ (Note: Ω still bounded)

<u>Pf</u> Sum of harmonic function and subharmonic fn. is subharmonic. Apply the theorem.