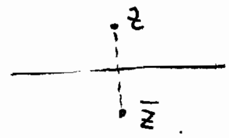
11-10-2010. Complex Analysis.

Poisson Kernal $P_z(\xi) = -\dfrac{\partial G_z(\xi)}{\partial n_\xi}$.

$-\ln|\xi - z| + \ln|\xi - \bar{z}|$.



- $u \in C^2(\Omega)$, Then $u \in SH(\Omega) \Leftrightarrow \Delta u \geq 0$.

Pf. $\displaystyle\int_{\mathbb{T}} u(z) \frac{|dz|}{2\pi} - u(0) = \frac{1}{2\pi} \iint_{\mathbb{D}} \Delta u(z) \underbrace{\ln \frac{1}{|z|}}_{\geq 0} dA(z)$.

We prove it in unit disc, then we can have the conclusion for $\forall$disc.

$$\int_{|z-z_0|=r} u(z) \frac{|dz|}{2\pi r} - u(z_0) = \begin{cases} \geq 0 & \text{if } \Delta u \geq 0,\ |z-z_0| = r \\ < 0 & \text{if } \Delta u < 0. \end{cases}$$

(Note: If $\Delta u(z_0) < 0$, then $\Delta u < 0$ on a small neigh. of z_0).

- $u \in SH$ means $\Delta u \geq 0$ as disstr.

Non-negative disstr. are always measures.

$\Delta u =$ measure ≥ 0.

- $\log|z-a| \in SH$.

$u(z) = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{C}} \ln|z - w|\, d\mu(w)$.

If $d\mu(w) = F(w)\, dA(w)$. $F \geq 0$, $F \in C_{comp}$. Then $\Delta u(z) = F(z)$.

So. If μ is finite, compactly supported measure on $\mathbb{C}$.

and $u(z) = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{C}} \ln|z-w|\, d\mu(w)$. $\leftarrow$ upper semi-continuous.

Then $\Delta u = \mu$. (in the sence of disstribution)

- Perron Process.

Ω - "bounded" region.

Given $h \in C(\partial\Omega)$, To find $u \in \mathrm{Harm}(\Omega) \cap C(\mathrm{cl}\,\Omega)$ s.t. $u|_{\partial\Omega} = h$.

real valued.

This is called Perron process

Solution:

Let $\mathcal{F}(h) = \{ v \in SH(\Omega) : \forall \xi \in \partial\Omega,\ \limsup_{z\to\xi} v(z) \le h(\xi) \}$.

real valued.

Define $u(z) = \sup\{ v(z) : v \in \mathcal{F}(h) \}$.

Observe: If $M = \max_{\partial\Omega} h$, then $\limsup_{z\to\xi} v(z) \le M,\ \forall \xi \in \partial\Omega.\ \forall v \in \mathcal{F}(h)$.

$\Rightarrow v(z) \le M \Rightarrow u(z) \le M$. So $u(z)$ is a (finite) function.

Thm. $u(z)$ is harmonic in Ω.

Lemma 1. If $v_1, v_2 \in SH$, $w(z) = \max\{v_1(z), v_2(z)\}$. Then $w \in SH$.

Pf. Fix z_0. WLOG. $w(z_0) = v_1(z_0)$.

Then $w(z_0) = v_1(z_0) \le \frac{1}{2\pi r} \int_{|z-z_0|=r} v_1(z)\,|dz|$, $\forall r < r_0$

$$\le \frac{1}{2\pi r} \int_{|z-z_0|=r} w(z)\,|dz| .$$

(Rmk. max of upper semi-continuous functions is upper-semi-cont.)

Lemma 2. Let $v \in SH(\Omega)$, $D \subset\subset \Omega$ ($\mathrm{cl}D \subset \Omega$). Let $\tilde{v}(z) = \begin{cases} v(z) & z \notin D \\ \mathcal{H}v|_{\partial D} & \text{if } z \in D. \end{cases}$

Then $\tilde{v}(z)$ is subharmonic.

Pf. $\tilde{v}$ is U.S.C. & SH in Ext D.

$\tilde{v}$ is harmonic in D. Only need to check $\tilde{v}$ is USC. and satisfy wMVI$_1$ on ∂D.

Consider $v_n \in C(\mathrm{Cl}\,D) \cap \mathrm{Harm}(D)$, $v_n \searrow \tilde{v}$.

(Take $v_n|_{\partial D} \searrow v|_{\partial D}$. then take harmonic extension.)

$\Rightarrow \tilde{v} \in \mathrm{USC}$ in $\mathrm{cl}\,D$ and $\tilde{v} \in \mathrm{USC}$. in $\Omega \setminus D$.

$\Rightarrow \hat{v} \in \mathrm{USC}$. Also $\hat{v} \geq v$.

So $\forall z_0 \in \partial D$, $\frac{1}{2\pi r} \int_{|z-z_0|=r} \hat{v}(z)\,|dz| \geq \int_{|z-z_0|=r} v(z) \frac{|dz|}{2\pi r} \geq v(z_0) = \hat{v}(z_0)$.

Take $\sigma_n \in C(\partial D)$, $\sigma_n \searrow \sigma$

and take their harmonic (Poisson) extension σ_n

$$\sigma_n(z) = \int_{\partial D} P_z^D(\xi)\, \sigma_n(\xi) \frac{|d\xi|}{2\pi}$$

where $P_z^D(\xi)$ is the Poisson kernel for D

(we use the same letter for the function σ_n and its Poisson extension)

Then by Monotone Convergence, because $\sigma_n \searrow \sigma$ on ∂D

$$\lim \sigma_n(z) = \int_{\partial D} P_z^D(\xi)\, \sigma(\xi) \frac{|d\xi|}{2\pi} = \tilde{\sigma}(z) \quad z \in \mathbf{D}$$

Note, that by maximum strong principle

$\sigma \leq \tilde{\sigma}$ in $\mathbf{D}$