COMPLEX ANALYSIS 11/12/10

$F(h) = \{ v \in SH(\Omega) : \forall \xi \in \partial\Omega \ \limsup_{z \to \xi} v(z) \leq h(\xi) \}$.

1) If $v_1, v_2 \in F(h)$ then $\max(v_1, v_2) := v_1 \vee v_2 \in F(h)$

2) If $v \in F(h)$ on $D \subset\subset \Omega$

let $\tilde{v}(z) = \begin{cases} v(z) & z \notin D \\ Hv(z) & z \in D \end{cases}$ then $\tilde{v}(z) \in F(h)$

Theorem:

$u(z) = \sup \{ v(z) : v \in F(h) \}$ then u is harmonic.

Proof:

Take $z_0 \in \Omega$ Take $v_n \in F$ st $v_n(z_0) \to u(z_0)$

let $v_n^1 = v_1 \vee v_2 \ldots \vee v_n$ so $v_n^1 \uparrow$ and $v_n^1 \geq v_n$.

$u(z_0) \leq \lim v_n(z_0) \leq \lim v_n^1(z_0) \leq u(z_0)$.

$D = D(z_0, r) \subset\subset \Omega$

$v_n^2(z) = \begin{cases} v_n^1(z) & z \notin D \\ H v_n^1(z) & z \in D \end{cases}$

$v_n^2 \uparrow$, $v_n^2 \geq v_n^1 \geq v_n$, v_n^2 is Harmonic in D.

$\lim v_n^2(z_0) = u(z_0)$

$v(z) = \lim_{n \to \infty} v_n^2(z)$ harmonic in D.

Limit of increasing sequence of harmonic functions is harmonic

In MVP, interchange limit and $\int$ by monotone convergence

+ <u>Want to show that $v(z) = u(z) \quad \forall z \in D$</u>.

Take arbitrary $z_1 \in D$, $w_n \in F(h)$, $\lim w_n(z_1) = u(z_1)$

$w_n^1 = w_n \vee v_n^2$

$w_n^2 = w_1^1 \vee w_2^1 \vee \dots \vee w_n^1$.

w_n^2 increasing and $w_n^2 \geq v_n^2$

$$w_n^3 = \begin{cases} w_n^2(z) & z \notin D \\ H w_n^2(z) & z \in D. \end{cases}$$

$w_n^3 \nearrow$, $w_n^3 \geq v_n^2$.

$\lim w_n^3(z_0) = u(z_0)$

$\lim w_n^3(z_1) = u(z_1)$

$w(z) = \lim w_n^3(z)$

w harmonic in D.

$w - v$ harmonic in D, ≥ 0 in D

$w(z_0) - v(z_0) = 0$.

So by Minimum Principle. $w(z) \equiv v(z)$

$u(z_1) = w(z_1) = \cancel{\lim}\, v(z_1) \quad \forall z_1 \in D$. $\square$

<u>Defn:</u>

Let $\xi \in \partial\Omega$. We say there is a barrier at ξ if $\forall \delta > 0$

$\exists\, b = b_{\xi,\delta}$ st.

1) $-b$ is subharmonic in Ω

2) $b \geq 0$.

3) $\lim_{z \to \xi} b(z) = 0$

4) $\forall z \in \Omega \ |z-\xi| > \delta \Rightarrow b(z) \geq 1$.

<u>Theorem</u>:

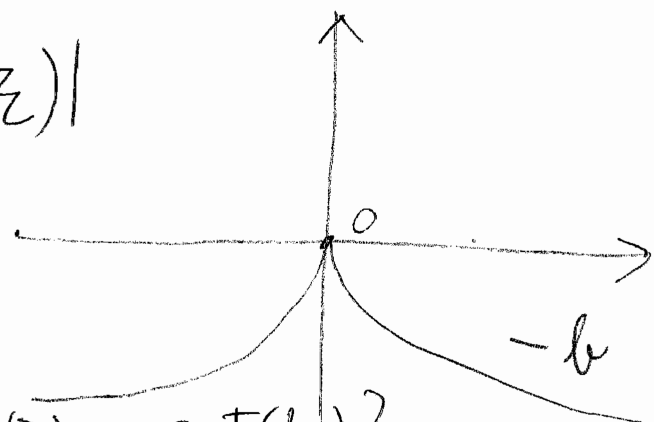
Let there be a barrier at each point $\xi \in \partial\Omega$

Then $\forall h \in C(\partial\Omega)$, $\exists u \in Harm(\Omega) \cap C(cl\,\Omega)$

st $u|_{\partial\Omega} = h$.

<u>Remark</u>: If Dirichlet Problem is solvable then $\exists$ a barrier at each point.

Solve DP for $h(\eta) = |(\eta - \xi)|$



<u>Theorem</u>:

let $\xi \in \partial\Omega$ and let there be a barrier at ξ. Let $u(z) = \sup\{v(z) : v \in F(h)\}$

then $\lim_{z \to \xi} v(z) = h(\xi)$.