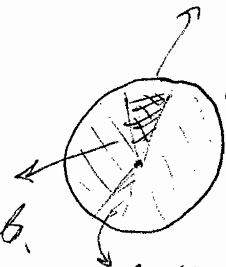
Notes on HW $\log\frac{1}{|z|} = \int\limits_{|z|=r} u(z)\frac{|dz|}{r} - \ln r \int\limits_{|z|=r} \frac{\partial u}{\partial n}|dz|$ | 11/15/2010

THIS IS CONSTANT

THEN APPLY GREEN'S FUNCTION WITH $\ln\left|\frac{1-\bar{z}_0 z}{z-z_0}\right|$. Make these agree.

<u>ANOTHER WAY</u>: Given u, v, $u+iv$ Hol.

∃ Harm conjugate



∃ Harm conjugate.

Doubly valued in this region, but they differ by 2π.

To fix this, let $a\alpha = 2\pi$. Then $f = \exp(a(u+iv))$ is Hol. and well defined.

This is our bounded analytic function, $\neq 0$.

Now take log, then real part, then divide by a for our result.

<u>TH</u>: For $\xi \in \partial\Omega$, there is a barrier at ξ.

$u(z) := \sup\{v(z) : v \in \mathcal{F}(h)\}$. Then $\lim\limits_{z\to\xi} u(z) = h(z)$, for h continuous

<u>PROOF</u>: Take $\varepsilon > 0$, then $\exists \delta > 0$, $\forall \eta \in \partial\Omega$, $|\xi - \eta| < \delta \Rightarrow |h(\xi) - h(\eta)| < \varepsilon$.

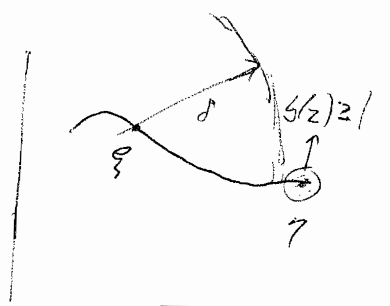
Let $M = \max\limits_{\partial\Omega} |h(\eta)|$, $s(z) = h(\xi) - \varepsilon - 2b(z).M$ → BARRIER

So $\lim\limits_{z\to\xi} s(z) = h(\xi) - \varepsilon$. So $s \in \mathcal{F}(h)$, since $\overline{\lim\limits_{z\to\eta}}\, s(z) \le h(\eta)$ $\forall \eta \in \partial\Omega$ $(*)$

To see this, note that if $|\eta - \xi| \le \delta$, then $|h(\eta) - h(\xi)| < \varepsilon \Rightarrow h(\eta) > h(\xi) - \varepsilon$.

But $s(z) \le h(\xi) - \varepsilon < h(\eta)$, so $(*)$ holds for $|\eta - \xi| \le \delta$.

If $|\eta - \xi| > \delta$. $\overline{\lim\limits_{z\to\eta}}\, s(z) \le h(\xi) - \varepsilon - 2M \le h(\eta)$



Then if $s \in \mathcal{F}(h) \Rightarrow u \ge s$, so $\underline{\lim\limits_{z\to\xi}}\, u(z) \ge h(\xi) - \varepsilon$. Take $\varepsilon \to 0$

Thus $\underline{\lim\limits_{\substack{z\to\xi \\ z\in\Omega}}}\, u_h(z) \ge h(\xi)$, $\underline{\lim\limits_{\substack{z\to\xi \\ z\in\Omega}}}\, u_{-h}(z) \ge -h(\xi) \Rightarrow \overline{\lim\limits_{z\to\xi}}(-u_{-h}(z)) \le h(\xi)$

If we show $u_h(z) + u_{-h}(z) \le 0$ $(+)$, we are done $\left(\overline{\lim\limits_{z\to\xi}}\, u_h(z) \le \overline{\lim\limits_{z\to\xi}} - u_{-h}(z) \le h(\xi)\right)$.

We now show (†). $\longrightarrow$ Take $v_1 \in \mathcal{F}(h)$, $v_2 \in \mathcal{F}(-h)$. Then $\varlimsup_{z\to\xi}(v_1(z)+v_2(z)) \le \varlimsup_{z\to\xi} v_1(z) + \varlimsup_{z\to\xi} v_2(z)$

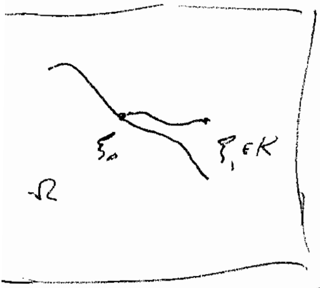
$\le h(\xi)+(-h(\xi))=0.$

By Maximum Modulus Principle, $v_1(z)+v_2(z)\le 0$. Taking sup gives $u_h(z)+u_{-h}(z)\le 0$. ☒

TH: Let $\xi \in \partial\Omega$ where $\exists$ connected compact $K \in \Omega^c$. $K \ne \{\xi\}$ s.t. $\xi \in K$.

Then $\exists$ a barrier at ξ.

PROOF: Let $\xi = \xi_0$, $\varphi_1(z) = \dfrac{z-\xi_1}{z-\xi_0}$ $\varphi_1(K) = K_1$.



A Trick: $\forall \delta>0, \exists R$ s.t. if $\left|\dfrac{z-\xi_1}{z-\xi_0}\right| > R \Rightarrow |\xi_0 - z| < \delta$

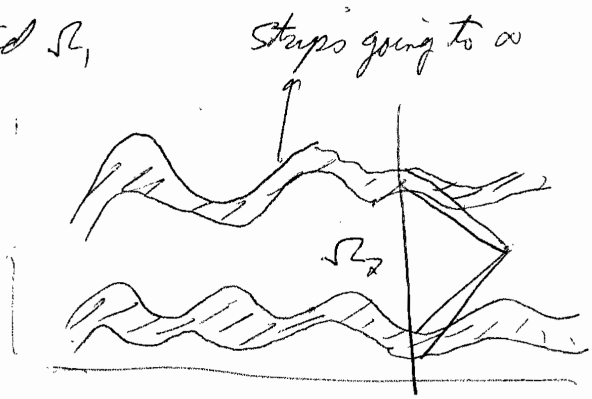
Look for $G(z) \le -1$ if $|z|\le R$ and $\lim_{|z|\to\infty} G(z) = 0$ G subharm, ≤ 0.

Need to define G in component of $\hat{\mathbb{C}}\setminus K$, containing ∞, denoted Ω_1

$\exists$ Branch of $\log(z)$ in Ω_1 $\Omega_2 = \log(\Omega_1)$.

Take $\Omega_2 \cap x_0 + i\mathbb{R} = \bigcup(a_k, b_k)$

open set. Take $x_0 \ge \ln R$.



Note: $\sum |a_k - b_k| \le 2\pi$, (Take exp of $\log(\Omega_1)$).

Let $G(z) = \begin{cases} \left(\sum \operatorname{Arg}(z-b_k) - \operatorname{Arg}(z-a_k)\right)/\pi & 0 \le \arg < 2\pi \quad \operatorname{Re}(z) \ge x_0 \\ -1 & \operatorname{Re} z \le x_0. \end{cases}$

When we translate $G(z)$ back, we obtain our desired barrier.

Remark Branch of $\log z$ exists in every connected component of ~~K_1 be~~ $\mathbb{C}\setminus K_1^{\,\prime}$ because any closed curve has index zero w/respect to any pt of K (Index w/respect to a depends continuously on the point a and $\operatorname{Ind}_a \gamma = 0$ for suff large a)