Cor If all connected components of Ω^c are non-trivial (not single pt)
Then Dirichlet problem is solvable in Ω.

Harmonic measure

Let Ω be regular domain, i.e. $\exists$ barrier at all $\xi \in \partial\Omega$,

Then $\forall\, h \in C(\partial\Omega)$

$$\exists!\ u = u_h \,, \qquad u \in Harm(\Omega) \cap C(Cl\,\Omega)$$

$$u_h\big|_{\partial\Omega} = h$$

Fix $z_0 \in \Omega$, $\quad \underset{C(\partial\Omega)}{\overset{h}{\cap}} \longmapsto u_h(z_0)$

By Max Principle, $\quad |u_h(z_0)| \leq \|h\|_\infty$

$h \geq 0 \;\Rightarrow\; u_h \geq 0 \;\Rightarrow\; u_h(z_0) \geq 0$

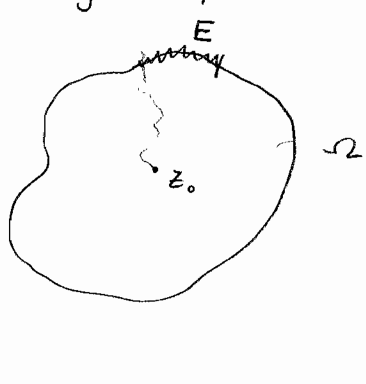
By Riez Representation TH

$\exists!$ Radon measure $\quad \omega = \omega^{\Omega}_{z_0}$, supported on $\partial\Omega$

$$\text{s.t.} \quad u_h(z_0) = \int_{\partial\Omega} h(\xi)\, d\omega(\xi)$$

Def: $\omega^{\Omega}_{z_0}$ is called Harmonic measure.

Probability Interpretation



$\omega^{\Omega}_{z_0}(E) = \text{Prob}($ Brownian motion started at z_0 exits $E\,\}$

Note. $\omega^{\Omega}_{z_0}(\partial\Omega) = 1 \qquad (h \equiv 1)$

Eg: 1) $\Omega = \mathbb{D}$

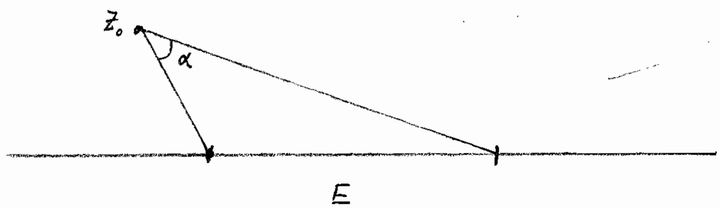
$$d\omega_{z_0}^{\mathbb{D}} = \frac{1-|z_0|^2}{|1-\bar{z}_0 z|^2} \frac{|dz|}{2\pi}$$

2) $\Omega = \mathbb{C}_+$

$$d\omega_{z_0}^{\mathbb{C}_+} = \frac{1}{\pi} \frac{\operatorname{Im}(z_0)}{|x - z_0|^2} dx$$

$$\omega_z^{\mathbb{C}_+}(E) = (\text{aperture of } E \text{ from } z_0)/\pi = \frac{\alpha}{\pi}$$

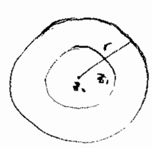
(obtained by direct computation $\int_E \frac{1}{\pi} \frac{y_0}{(x-x_0)^2 + y_0^2} dx$)



Prop: Let $z_0, z_1 \in \Omega$. $\exists\, c, C$ depending on $z_0, z_1 \in \Omega$

s.t. $c\,\omega_{z_1}^{\Omega} \leq \omega_{z_0}^{\Omega} \leq C\,\omega_{z_1}^{\Omega}$

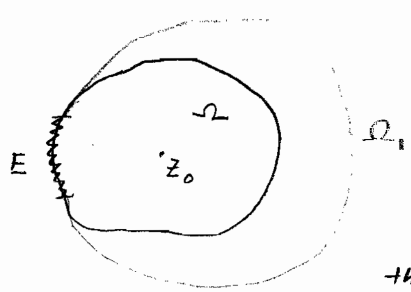
Follows from Harnak inequality: u positive harmonic.



$$|z_1 - z_0| \leq \frac{r}{2}$$

$$u(z_0)\frac{1-\frac{1}{2}}{1+\frac{1}{2}} \leq u(z_1) \leq \frac{1+\frac{1}{2}}{1-\frac{1}{2}} u(z_0)$$

Monotonicity of harmonic measure.



$\Omega \subset \Omega_1$

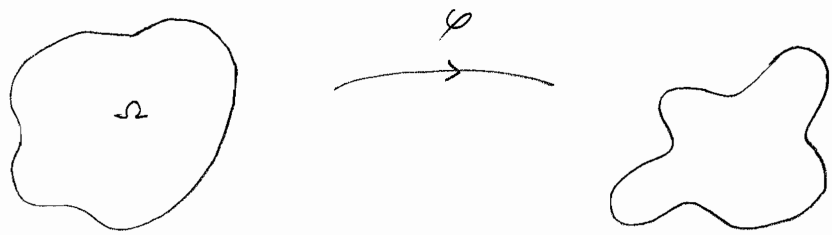
$E \subset \partial\Omega \cap \partial\Omega_1$

then $\omega_{z_0}^{\Omega}(E) \leq \omega_{z_0}^{\Omega_1}(E)$

$E \Subset \partial\Omega \cap \partial\Omega_1$, $\exists\ c, C$ depend on Ω, Ω_1, z_0, E s.t

$$c\,\omega_{z_0}^{\Omega_1}\big|_E \lesssim \omega_{z_0}^{\Omega}\big|_E \leq C\,\omega_{z_0}^{\Omega_1}\big|_E \qquad \text{on } E.$$

Prop: Harmonic measure is conformally invariant



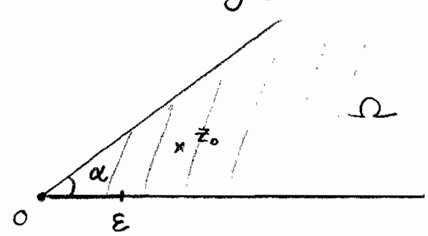
Suppose

$$\varphi \in C(\mathrm{Cl}\,\Omega), \quad 1\text{-}1 \text{ on } \partial\Omega, \quad \varphi \text{ conformal on } \Omega$$

then

$$\omega_{z_0}^{\Omega}(E) = \omega_{\varphi(z_0)}^{\varphi(\Omega)}(\varphi(E))$$

Harmonic measure at an angle



$$\omega_{z_0}^{\Omega}(0,\varepsilon) = \ ?$$

$$\varphi : \Omega \longrightarrow \mathbb{C}_+ \quad \text{upper half plane.}$$
$$z \longmapsto z^{\frac{\pi}{\alpha}}$$

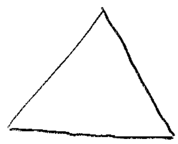
$$\varphi(0,\varepsilon) = (0, \varepsilon^{\frac{\pi}{\alpha}}) \quad , \quad \text{then } \omega_{z_0}^{\Omega}(0,\varepsilon) \sim \varepsilon^{\frac{\pi}{\alpha}}$$

TH (N. Makarov, P. Jones)

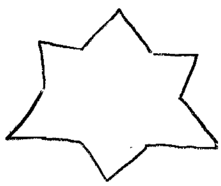
$\forall$ regular Ω, $\omega_{z_0}^{\Omega}$ is supported on a set of Hausdorff dim ≤ 1

Remark: Hausdorff dim $\partial\Omega$ can be > 1

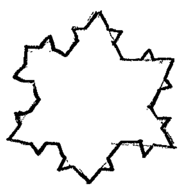
Eg: snow flake:



Ω_1



Ω_2



Ω_3

$\vdots$

$\dim \partial \Omega_\infty > 1$