Green's Function

$\exists$ barrier at every point

Let Ω be a regular domain.

Let u be the solution to the Dirichlet Problem on Ω with boundary values $\ln|z-a|$, $a \in \Omega$, $z \in \partial\Omega$.

We can define $G_a(z) = u(z) - \ln|z-a|$.

So $G_a(z) = 0 \quad \forall z \in \partial\Omega$,

and $G(z) = \ln \frac{1}{|z-a|} + \underset{\text{function}}{\text{bdd}}$ near a.

Let Ω be a simply connected domain with $\hat{\mathbb{C}} - \Omega \neq \emptyset, \{p\}$, so Ω is regular.

Define $\varphi(z) = (z-a)\, e^{-(u+i\tilde{u})}$ ($\sim$ means conjugate)

$= \text{"} e^{-(G_a + i\tilde{G}_a)} \text{"}$ (region is simply connected, so the conjugate is well-defined.)

We put this in quotes because this is symbolic notation. $\tilde{G}_a$ is not single-valued.

<u>Claim</u>: φ is a conformal map $\Omega \to \mathbb{D}$.

<u>Pf</u>/ $|\varphi(z)| = e^{-G_a(z)} \leq 1$ since $G_a \geq 0$.

By construction, $\varphi(z) = 0 \iff z = a$.

Take w_0 in the unit disc, so $|w_0| < 1$. We want to show that the equation $\varphi(z) = w_0$ has a unique solution.

Consider $\Omega_r = \{ z \in \Omega : |\varphi(z)| = e^{-G_a(z)} < r \}$.

Note that $\bigcup \Omega_r = \Omega$ and $\partial\Omega_r = \{ z \in \Omega : |\varphi(z)| = r \}$.

In fact, $\partial\Omega_r$ is a piecewise-C^1 curve, so Rouché's Thm applies.

Now, if $r > |w_0|$, then $|w_0| < |\varphi(z)|$ on $\partial\Omega_r$.

Therefore, $\varphi(z) = 0$ and

$\varphi(z) - w_0 = 0$

have the same number of solutions in Ω_r by Rouché's Thm, —

i.e. both have 1 solution.

Note that this is an alternate proof of the Riemann Mapping Theorem.

Approximation by Rational Functions

i.e. $f \in Hol(\Omega)$ for some open set $\Omega \supset K$.

Runge's Thm/ Let $K \subset \mathbb{C}$ be compact, $f \in Hol(K)$. Then f can be uniformly approximated by rational functions with poles in K^c.

$\hookrightarrow$ in $\|\cdot\|_{C(K)}$

Moreover, if we fix one point in each connected component of K^c, (a_k), then we can require the rational functions to have poles only at (a_k).

Pf/ Cover K by finitely many discs D_k, with each $D_k \Subset \Omega$.

Let $G = \bigcup_k D_k$, so ∂G is piecewise-C^1 (in fact piecewise-analytic), and ∂G is homologous to zero. Let $\gamma = \partial G$, so by Cauchy's Theorem:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi$$

Let us split $\gamma = \bigcup_k \gamma_k$ (disjoint), where $\gamma_k \in D(\xi_k; r_k) \Subset K^c$.

Let $f_k(z) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(\xi)}{\xi - z}\, d\xi$, so $f(z) = \sum_k f_k(z)$ (this is a finite sum).

Note that $f_k \in Hol(\gamma_k^c) \subset Hol(D(\xi_k; r_k)^c)$, so we have the Laurent expansion

$$f_k(z) = \sum_{n=0}^{\infty} C_n^k (z - \xi_k)^{-n},$$ for some constants C_n^k.

Since $\lim_{z \to \infty} f_k(z) = 0$, we know that the Laurent expansion does not have any nonnegative terms.

Also, convergence is uniform on compact subsets, so in particular, we have uniform convergence on K.

This proves the first part of the theorem.

To prove the second part, the following lemma is required to "shift the poles":

<u>Lemma (shift of the poles)</u>/ Let U be open, connected, $a \in U$. Then any rational function with poles in U can be approximated (uniformly on U^c) by rational functions with poles at a.

<u>Idea</u>:

If z_1 is "close" to z_0, — say z_0 is contained in a disc at z_1 which is cpctly supported in the domain

then $\forall k \geq 0$, $\frac{1}{(z-z_0)^k}$ can be approximated by $\sum_0^N \frac{c_n}{(z-z_1)^n}$. (Use power series.)

By continuous induction, we can show that z_1 can be moved to any point

Details are left as a homework assignment.

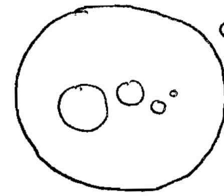
<u>Mergelyan's Thm</u>/ If K is simply connected, compact, then any $f \in C(K) \cap \mathrm{Hol}(\mathrm{Int}\, K)$ can be uniformly approximated by polynomials.

(Polynomials are just rational functions with poles at ∞.)

This proof is rather non-trivial; we will not prove this theorem.

* This theorem is not true for rational approximation. (<u>if we throw out the assumption of simply connectedness</u>.)

(Swiss Cheese)



$Cl(\mathbb{D}) - \bigcup D_k$

$\Omega = \mathbb{D} - \underbrace{\bigcup D_k}$ the D_k that contain poles.

Example: Let $K = Cl(\mathbb{D}) - \bigcup D_k$,

where $D_k = D(z_k; r_k)$

are such that: ① $\{z_k\}$ is dense in K

② $\sum r_k^2 < 1$

③ $D_k \cap D_j = \emptyset$ for $j \neq k$.

If f is a rational function with poles in K^c, then

$$\int_{\mathbb{T}} f(z)\,dz - \sum \int_{\partial D_k} f(z)\,dz = 0.$$

If $f(z) = \bar{z}$, then

$$\int_{\mathbb{T}} \bar{z}\,dz - \sum \int_{\partial D_k} \bar{z}\,dz = 2\pi i - 2\pi i \underbrace{\sum r_k^2}_{<1} \neq 0.$$

But condition ① implies that $\operatorname{Int} K = \emptyset$.

So $f(z) = \bar{z}$ belongs to $C(K) \cap Hol(\operatorname{Int} K)$
$= C(K)$

but cannot be approximated by rational functions