Nov 29, 10

Infinite Products

$P = \prod_{k=1}^{\infty} p_k \quad , \quad p_k \neq 0$

partial product $P_n = \prod_{k=1}^{n} p_k$. Then $P = \lim_{n\to\infty} P_n$ if lim exists and finite ($\neq 0$).

If $p_n = p_n(z)$ analytic, $p_n(z) \neq 0$ in connected Ω. and $P(z_0) = \prod_{n=1}^{\infty} p_n(z_0) = 0 \Rightarrow P(z) \equiv 0$.

It's often convenient to write $p_n = 1 + a_n$.

① $\prod_{n=1}^{\infty} (1+a_n)$ ② $\sum_{n=1}^{\infty} \log(1+a_n)$.

If $(1+a_n) > 0$, convergent of ① is equivalent to convergent of ②.

Principal branch of log $\sim$ $\pi < \operatorname{Im} \log z \leq \pi$.

Thm If $a_n \in \mathbb{C}$, $a_n \neq -1$ for all n, then ① converges if and only if ② converges.

Pf. Let $P = \prod^{\infty} (1+a_n) \neq 0$.

If $P_n = \prod_{k=1}^{n} (1+a_k)$, $P = \lim_{n\to\infty} P_n$, so $\lim_{n\to\infty} P/P_n = 1$.

$\lim_{n\to\infty} \frac{P_n}{P_{n-1}} = \lim_{n\to\infty} \frac{P_n}{P} \cdot \frac{P}{P_{n-1}} = 1$

$\quad \|$

$\lim (1+a_n)$

Thus, $\lim_{n\to\infty} a_n = 0$.

So, $\exists N$ s.t. $\left| \frac{P}{P_N} - 1 \right| < \frac{1}{10}$, $\forall n \geq N$, $|a_n| < \frac{1}{10}$.

Let $\tilde{P} = P/P_N = \prod_{n=1}^{\infty} (1 + a_{N+n})$. This product is closed to 1.

$\tilde{P}_n = \prod_{k=1}^{n} (1 + a_{N+n})$, $\exists M$ s.t. ... satisfies $|\tilde{P}_n - 1| < \frac{1}{5}$, $\forall n \geq M$.

Now, take log to both sides and we're done.

and.

$\lim_{n\to\infty} \tilde{P}_n$ exists ^ $\neq 0$.

So, $\lim_{n\to\infty} \log \tilde{P}_n$ exists

If $n > M$ then $|\tilde{P}_n - 1| < \frac{1}{5}$.

$\Rightarrow \log \tilde{P}_n / \tilde{P}_{n-1} = \log(1 + a_{n+N})$. $\square$

<u>Def</u> $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely if $\sum_{n=1}^{\infty} |\log(1+a_n)| < \infty$.

absolute convergent $\to$ rearrangement doesn't change limit.

$\lim_{z\to 0} \frac{\log(1+z)}{z} = 1$. So, by limit comparison test, $\sum_{n=1}^{\infty} |\log(1+a_n)| < \infty \Leftrightarrow \sum_{n=1}^{\infty} |a_n| < \infty$.

Let $f_n \in Hol(\Omega)$.

$P_n = \prod_{k=1}^{n} f_k(z)$

$P_n(z) \to P(z) \neq 0$.

$\forall K \subset \Omega$, K compact, $P(z)$ has finitely many zeros in K. $\Rightarrow$ Only finitely many f_n have zero on K.

$\left\{ \begin{array}{l} \text{If } m \text{ of } f_n \text{ have zero at } z_0 \Rightarrow P^{(m-1)}(z_0) = 0. \\ \text{. } P \text{ has zero of multiplicity at least } m \text{ at } z_0. \end{array} \right\}$

Q: $\sum \log(1+a_n)$ $\qquad$ $\sum a_n$.

Construct a series when one of the series converges while the other one is not.

(enough to have $a_n \in \mathbb{R}$).

Blaschke Products

Let $\alpha_k \in \mathbb{D}$, $\alpha_k \neq 0$. Consider $\displaystyle\prod_{k=1}^{\infty} \underbrace{\frac{|\alpha_k|}{\alpha_k} \frac{\alpha_k - z}{1 - \bar{\alpha}_k z}}_{b_{\alpha_k}(z).}$.

$b_{\alpha_k}(0) = |\alpha_k| > 0.$

If $z=0$, then we have $\prod |\alpha_k|$ which converges

$$\Updownarrow$$

$\sum \log|\alpha_k|$ converges

$$\Updownarrow$$

$\sum \log|\alpha_k|$ converges absolutely

$$\Updownarrow$$

$$\sum (1-|\alpha_k|) < \infty.$$

$\to$ necessary condition for convergence of Blaschke producti

Sufficient

$$|1 - b_{\alpha_k}(z)| \leq C(z)\,(1-|\alpha_k|)$$