Lecture # 33

Last lecture: Mo, 6. Dec.

In class exam: Fr, 10. Dec.

Zeroes of analytic functions

$H^\infty(\mathbb{D})$:= bounded analytic in $\mathbb{D}$

Any sequence $a_n \in D$, s.t. $\sum(1-|a_n|) < \infty$ is a zero sequence (counting multiplicities) of a $f \in H^\infty$

$$B = \prod b_{a_n}$$

$$b_a := \frac{|a|}{a} \quad \frac{a-z}{1-\bar{a}z} \qquad \text{Formally } b_0 = z$$

$\boxed{\text{Thm.:}}$ Let $f \in H^\infty(\mathbb{D})$ and let a_n be zeroes of f (counting multiplicities) Then $\sum(1-|a_n|) < \infty$

$\boxed{\text{Lemma}}$ (Jensen's formula)

Let $f \in Hol(\overline{D}_R)$, $f(0) \neq 0$. Then

$$\ln|f(0)| = \int_0^{2\pi} \ln|f(Re^{it})| \frac{dt}{2\pi} + \sum \ln \frac{|a_n|}{R}$$

Pf.: Note:
$\tilde{f}(z) = f(Rz) \in Hol(\overline{\mathbb{D}})$ with zeroes $\frac{a_n}{R}$

So we only need to consider $R=1$.

Assume $f(z) \neq 0$ $\forall z$ with $|z| = R$.

$$B(z) = \prod b_{a_n}(z)$$

is a finite product since $B(z)$ has only finitely many zeroes in a compact set

Factorize

$$f = B \cdot g \quad , \quad g(z) \neq 0 \quad \forall z \in \overline{\mathbb{D}}$$

$$\ln |B(0)| = \sum \ln |a_k|$$

$$\ln |g(0)| = \int_0^{2\pi} \ln |g(e^{it})| \frac{dt}{2\pi}$$

$\log |g(z)| = \operatorname{Re} \log g(z)$ is therefore harmonic.

mean value property

$$= \int_0^{2\pi} \ln |f(e^{it})| \frac{dt}{2\pi}$$

$|B(e^{it})| = 1$

proved for $R = 1$ & so for all R, assuming that $f(z) \neq 0 \ \forall |z| = R$.

For the general case, we can take $R_n \longrightarrow R$ s.t. $f(z) \neq 0 \ \forall |z| = R_n$.

(use logarithmic singularity & dominated convergence) $\square$

<u>Pf. of thm:</u> Let $f \in H^\infty$, a_n zeroes of f. $R < 1$

$$\sum_{|a_n| < R} \ln \frac{R}{|a_n|} = -\ln |f(0)| + \underbrace{\int_0^{2\pi} \ln |f(Re^{it})| \frac{dt}{2\pi}}_{\leq C}$$

$$\sum_{|a_n| < R} \ln \frac{R}{|a_n|} \leq C$$

Let $R \longrightarrow 1$, then by monotone convergence

$$\sum_{|a_n| < R} \ln \frac{1}{|a_n|} \leq C$$

So $\sum \ln \frac{1}{|a_n|} < \infty$

$$\Updownarrow \quad \text{limit comparison test}$$

$\sum (1-|a_n|) < \infty$ $\square$

$\boxed{\text{Thm:}}$ (Weierstrass)

Let $a_n \in \mathbb{C}$, $\lim_{n\to\infty} |a_n| = \infty$ then $\exists f \in \mathrm{Hol}(\mathbb{C})$ s.t.

a_n are zeroes of f (counting multiplicity)

Pf.: Assume $a_n \neq 0$ $\forall n$. Take

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{p_n(z/a_n)}$$

Taylor:

$$\ln(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}$$

since $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ & integration. $|z| < 1$

Set $p_n(z) := \sum_{k=1}^{n} \frac{z^k}{k}$. Assume $|z| \leq \frac{1}{2}$, then

$$|\ln(1-z) + p_n(z)| = \left| \sum_{k=n+1}^{\infty} \frac{z^k}{k} \right| \leq |z|^{n+1} \cdot \underbrace{\sum_{k=0}^{\infty} |z|^k}_{\leq 2} \leq 2|z|^{n+1}$$

Now

$$\sum \left[\ln\left(1 - \frac{z}{a_n}\right) - p_n\left(\frac{z}{a_n}\right) \right]$$

$|z| \leq R$: Consider only $|a_n| \geq 2R$

Each term is bounded by $2\left|\frac{z}{a_n}\right|^{n+1} \leq 2^{-n}$

Thus the above series is absolutely convergent. $\square$

Can we take p_n with fixed degree h?

$$\left| \ln\left(1 - \frac{z}{a_n}\right) + P_h\left(\frac{z}{a_n}\right) \right| \leq 2 \frac{|z|}{|a_n|^{h+1}}$$

$|z| \leq R$

$$\leq \frac{2R}{|a_n|^{h+1}}$$

$|a_n| \geq 2R$

Need condition

$$\sum \frac{1}{|a_n|^h} < \infty$$

$\rightsquigarrow$ Hadamard factorization