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Multivalued Functions, Riemann Surfaces, & Analytic Extension

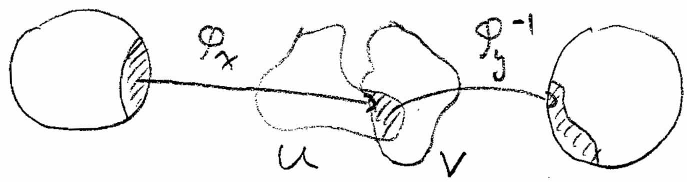
e.g. $\sqrt{z}$, or $\log(z)$

Riemann Surfaces: Elementary approach

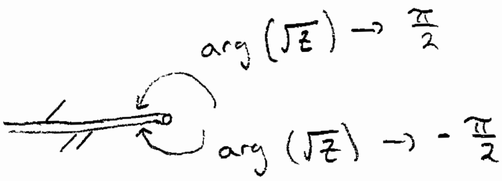
A Riemann surface is a $\mathbb{C}$-manifold of complex dimension 1.

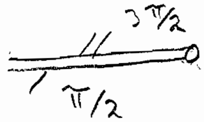
i.e. S s.t. $\forall x \in S$ $\exists$ nbhd $\mathcal{U} \ni x$ and a chart $\varphi_x : \mathbb{D} \xrightarrow[\text{homeomorphi.}]{\text{bijection}} \mathcal{U}$

and $\mathcal{U} \ni x$, $V \ni y$, $\mathcal{U} \cap V \neq \emptyset$, then $\varphi_y^{-1} \circ \varphi_x$ is analytic.



Consider $\sqrt{z}$:





glue a second copy of $\mathbb{C}$ with the given identificatio

cut the complex plane along the negative Re axis

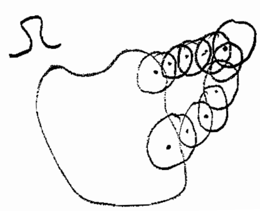
Similarly, for $\log z$: . . .

This beds nicely into $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ by $(z, \arg(z))$

Exercise. Construct the Riemann surface for $\sqrt{1-z^2}$

Analytic Continuation:

Let $f \in Hol(\Omega)$, if f is analytic on the given discs, we can extend it



Def: Analytic continuation along a path

Let $f \in Hol(\Omega)$, $z_0 \in \Omega$, path $\gamma : [a,b] \to \mathbb{C}$, $\gamma(a) = z_0$

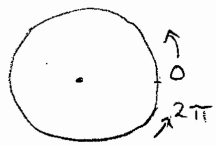
f admits analytic continuation along the path γ

if $\forall t \in [a,b]$, $\exists f_t \in Hol(V_t)$, V nbhd of $\gamma(t)$

such that $\forall t_1 \in [a,b]$, $\exists \delta > 0$ s.t $|t_1 - t_2| < \delta$

$\Rightarrow V_1 \cap V_2 \neq \emptyset$ & $f_{t_1} \equiv f_{t_2}$ on $V_1 \cap V_2$.

Example: $\log(z)$



Germs of analytic functions & Abstract construction of R.S.

Germ: $\xi \in \mathbb{C}$, $\mathcal{U}, V$ two nbhds of ξ $f_1 \in Hol(\mathcal{U})$, $f_2 \in Hol(V)$.

$f_1 \sim f_2$ if $\exists_{\xi \in} W \subseteq \mathcal{U} \cap V$ and $f_1 \equiv f_2$ on W.

Germ at ξ corresponds to an equivalence class f_ξ or (f, ξ)

$(f, \mathcal{U} \ni \xi)$ a representative

Let f_ξ, $(f, \mathcal{U})$ repr. Then $\forall \zeta \in \mathcal{U}$, define f_ζ, $(f, \mathcal{U})$

Sheaf of Germs $\mathfrak{S}(\mathbb{C})$ (gothis S)

Topology: Ω, $f_\xi \in \mathrm{int}(\Omega)$ if $\exists$ repr $f, \mathcal{U}$ such that $\forall \zeta \in \mathcal{U}$

$f_\zeta \in \Omega$

The RS. of a germ is just a connected component.