April 14, 2004

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Paradigm - All ONB are equivalent.

Examples: $A = A^* \Rightarrow$ e. values of $A^2$ are $\geq 0$ True

In some ONB matrix of $A$ is $D = \text{diag}(\lambda_1 \dots \lambda_n)$ $\lambda_k \in \mathbb{R}$

$A^2$, $D^2 = \text{diag}(\lambda_1^2 \dots \lambda_n^2)$ so e. values of $A^2 \geq 0$.

If $A^*A$ has only 0 & 1 as eigenvalues then $A$ is orthogonal projection. True.

$\vec{u}_1 \dots \vec{u}_n = \beta$ $[A]_{\beta\beta} = \begin{pmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & 1 & & \\ & 0 & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \}k$

If matrix of $N$ in some ONB is diagonal, then $N^*N = NN^*$

$N$-diag., $N^*$- diag. & diag. matrices commute.

$N = UDU^*$, $N^* = UD^*U^*$ $(U = U^{**})$, $NN^* = UDU^*UD^*U^* = UDD^*U^*$

$N^*N = UD^*U^*UDU^* = UD^*DU^*$ $D^*D = DD^*$

Def. $N: X \to X$ is called normal if $N^*N = NN^*$

Examples: 1. $A$ self adjoint $\Rightarrow$ $A$ normal $(A^*A = A^2 = AA^*)$

2. $U: X \to X$ unitary $\Rightarrow U$- normal $UU^* = I, U^*U = I$

Thm. (In $\mathbb{C}^n$) $N$-normal, then $\exists$ ONB $\vec{u}_1 \dots \vec{u}_n$ s.t. matrix of $N$ in this basis is diagonal $N = UDU^*$

Pf. Pick $\vec{u}_1 \dots \vec{u}_n$, ONB s.t. matrix of $N$ is upper triangular.

L. $N$-normal & upper triangular $\Rightarrow N$-diagonal.

Pf. of L.: Induction in $n$, 1x1 trivial

Assume true for $(n-1) \times (n-1)$. Prove that true for $n \times n$.

$N = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ 0 & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}$ $N^* = \begin{pmatrix} \overline{a_{11}} & & 0 \\ \overline{a_{12}} & \ddots & \\ \vdots & & \ddots \\ \overline{a_{1n}} & \cdots & \overline{a_{nn}} \end{pmatrix}$ $N^*N = \begin{pmatrix} |a_{11}|^2 & \\ & \\ & \end{pmatrix}$

$NN^* = \begin{pmatrix} |a_{11}|^2 + |a_{12}|^2 + \dots + |a_{1n}|^2 & \\ & \\ & \end{pmatrix}$ $a_{12} = a_{13} = \dots = a_{1n} = 0$

$N = \begin{pmatrix} a_{11} & 0 \dots 0 \\ 0 & \\ \vdots & N_1 \\ 0 & \end{pmatrix}$ upper triang. $N^*N = \begin{pmatrix} |a_{11}|^2 & 0 \dots 0 \\ 0 & \\ \vdots & N_1^*N \\ 0 & \end{pmatrix}$ $NN^* = \begin{pmatrix} |a_{11}|^2 & 0 \dots 0 \\ 0 & \\ \vdots & N_1N_1^* \\ 0 & \end{pmatrix}$

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$N_1$-normal & upper triang. $\Rightarrow N_1$- diagonal

<u>Ex.</u> $A=\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$ $\quad \begin{vmatrix} 1-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2+3^2=0 \quad 1-\lambda=\pm 3i, \ \lambda = 1\pm 3i$

$\lambda=1+3i: (A-\lambda I)=\begin{pmatrix} -3i & -3 \\ 3 & -3i \end{pmatrix} \quad \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \lambda=1-3i: (A-\lambda I)=\begin{pmatrix} 3i & -3 \\ 3 & 3i \end{pmatrix} \quad \begin{pmatrix} 1 \\ i \end{pmatrix}$

orthonormal: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$A=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} 1+3i & 0 \\ 0 & 1-3i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^* \qquad \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^* = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$

<u>Prop.</u> Any normal operator with real e. values is self adjoint.

<u>Pf.</u> - (diag. matrix with real entries is self adjoint)