Classnotes 4/16/04

Q A, B - self adjoint $\overset{?}{\Rightarrow}$ AB self adjoint False

$(AB)^* = AB$?

$(AB)^* = B^* A^* = BA \neq AB$

counterexample $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

Q A, B - normal $\overset{?}{\Rightarrow}$ AB normal False

$(AB)^*(AB) \overset{?}{=} (AB)AB^*$

$(AB)^*(AB) = B^*A^*AB = B^*AA^*B$

$(AB)(AB)^* = ABB^*A^* = AB^*BA^*$

Completion of OS to OB, $\vec{v}_1, \vec{v}_2 \dots \vec{v}_r$ - OS

$\exists\ \vec{v}_{r+1} \dots \vec{v}_n$ s.t. $\vec{v}_1 \vec{v}_2 \dots \vec{v}_n$ OB in X.

$E = \text{span}(\vec{v}_1 \dots \vec{v}_r)$ if $E = X$ - done if $E \neq X \Rightarrow \exists\ \vec{v}_{r+1} \perp E$, $\vec{v}_{r+1} \neq \vec{0}$

so $\vec{v}_1 \vec{v}_2 \dots \vec{v}_r \vec{v}_{r+1}$ OS

If $\text{span}(\vec{v}_1 \dots \vec{v}_{r+1}) = X$ - done

If not - increase system by one again, but cannot have more than n.

Practical Proof

$E = \text{span}(\vec{v}_1 \dots \vec{v}_r)$ add to $\vec{v}_1 \dots \vec{v}_r$ OB in $E^\perp$ $(\vec{v}_{r+1} \dots \vec{v}_n)$

(ex) $\mathbb{R}^3$ $\vec{v}_1 = (1, 2, 3)^T$ $E_1 = \text{span}\, \vec{v}_1$

$E^\perp = \text{Ker}\, \vec{v}_1^* = \text{Ker}\,(1\ 2\ 3)$ $x_2, x_3 \Rightarrow$ free $\begin{pmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix}$

$x_1 = -2x_2 - 3x_3$

basis in $E^\perp$

$x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ do gram-schmitt. $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} - \dfrac{\left(\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_2\right)}{\|\vec{v}_2\|^2} \vec{v}_2 =$

so $\vec{v}_3 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} - \frac{6}{5}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

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$$\boxed{\dim E + \dim E^{\perp} = n = \dim X}$$

<u>Def</u> $A = A^*$ is called positive definite $(A > 0)$ if $(A\vec{x}, \vec{x}) > 0$ $\forall \vec{x} \neq \vec{0}$. or is called positive semi definite $(A \geq 0)$ if $(A\vec{x}, \vec{x}) \geq 0$ $\forall \vec{x}$. or is called negative definite $(A < 0)$ if $(A\vec{x}, \vec{x}) < 0$ $\forall \vec{x} \neq 0$ or is called negative semidefinite $(A \leq 0)$ if $(A\vec{x}, \vec{x}) \leq 0$ $\forall \vec{x}$.

<u>Thm</u> Let $A = A^*$. ① A is positive definite iff all eigenvalues of A are > 0
② A is positive semidefinite iff all eigenvalues are ≥ 0

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if $A = \text{diagonal}(\lambda_1 \dots \lambda_n)$ $\lambda_k \geq 0$ then $\sqrt{A} = \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2} \dots \sqrt{\lambda_n})$

(Ex) A $m \times n$ then $A^* A \geq 0$

<u>Pf</u> $(A^* A \vec{x}, \vec{x}) = (A\vec{x}, \underset{A^{**}}{A}\vec{x}) = \|A\vec{x}\|^2 \geq 0$

<u>Def</u> $|A| = \sqrt{A^* A}$

$$\|A\vec{x}\| = \| \, |A| \vec{x} \, \|$$