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Singular Value Decomposition

A $\quad |A| = (A^*A)^{1/2}$ $\quad$ eigenvalues of $|A|$ are called singular values of A

$s_1, s_2 \dots s_n$

- If $\lambda_1 \dots \lambda_n$ are e.values of A^*A, then $s_k = \sqrt{\lambda_k}$

Ex $A = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$ $\quad A^*A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix}$

$$\begin{vmatrix} 8-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = (8-\lambda)(5-\lambda) - 4 = 40 - 13\lambda + \lambda^2 - 4$$
$$= \lambda^2 - 13\lambda + 36 = 0$$
$$(\lambda-4)(\lambda-9) = 0$$

$s_1 = 3 \quad s_2 = 2$

e.vectors of A^*A (also of A)

$\lambda = 9$ $\qquad\qquad \lambda = 4$

$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\frac{1}{\sqrt{5}}$ $\qquad \vec{x} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

A $\quad s_1 \dots s_n$ means A^*A has e.values $s_1^2, s_2^2, \dots s_n^2$

$\vec{v}_1 \dots \dots \vec{v}_n$ correspondin e.vectors ONB

Thm $\frac{1}{s_k} A\vec{v}_k = \vec{w}_k$ $\quad$ (k s.t. $s_k > 0$)

then $\vec{w}_k$ = ONS

Proof $(\vec{w}_k, \vec{w}_j) = \left(\frac{1}{s_k} A\vec{v}_k, \frac{1}{s_j} A\vec{v}_j\right) = \frac{1}{s_k s_j}(A\vec{v}_k, A\vec{v}_j)$

$$= \frac{1}{s_k s_j}(A^*A\vec{v}_k, \vec{v}_j)$$
$$= \frac{1}{s_k s_j}(s_k^2 \vec{v}_k, \vec{v}_j)$$
$$= \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

Cor $A\vec{v}_k = s_k \vec{w}$

Cor $A\vec{x} = \sum_{k=1}^{r} s_k (\vec{x}, \vec{v}_k)\vec{w}_k$ $\qquad$ $s_1 \dots s_r$ are non zero, singular values of A

$A = \sum_{k=1}^{r} s_k \vec{w}_k \vec{v}_k^*$ Singular Value Decomposition (SVD)

$\vec{w}_1 = \frac{1}{3} A\vec{v}_1 = \frac{1}{3}\frac{1}{\sqrt{5}}\begin{pmatrix}2 & -1\\ 2 & 2\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \frac{1}{3}\frac{1}{\sqrt{5}}\begin{pmatrix}3\\6\end{pmatrix}$

$\vec{w}_2 = \frac{1}{2}\frac{1}{\sqrt{5}}\begin{pmatrix}2 & -1\\ 2 & 2\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix} = \frac{1}{2}\frac{1}{\sqrt{5}}\begin{pmatrix}-4\\2\end{pmatrix}$

$A = 3\,\frac{1}{3\sqrt{5}}\begin{pmatrix}3\\6\end{pmatrix}\frac{1}{\sqrt{5}}(2,1) + 2\cdot\frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix}\frac{1}{\sqrt{5}}(-1,2)$

$$A = \underbrace{\begin{pmatrix}\frac{1}{\sqrt{5}} & -2/\sqrt{5}\\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}}_{W}\underbrace{\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}}_{\Sigma}\underbrace{\begin{pmatrix}2/\sqrt{5} & 1/\sqrt{5}\\ -1/\sqrt{5} & 2/\sqrt{5}\end{pmatrix}}_{V^*}$$

$V^*\vec{v}_1 = \begin{pmatrix}1\\0\end{pmatrix} = \vec{e}_1$

$\Sigma V^*\vec{v}_1 = 3\vec{e}_1$

$W\Sigma V^*\vec{v}_1 = 3\vec{w}_1 = A\vec{v}_1$

$A = W\Sigma V^*$ $\Rightarrow$ Σ = diagonal

W, V = unitary

<u>If</u> A $n\times n$, & all $\sigma_k < 0$

<u>Then</u> $V = [\vec{v}_1 \ldots v_n]$

$W = [\vec{w}_1 \ldots \vec{w}_r]$

$\vec{v}_1 \ldots \vec{v}_n$ $\mathcal{A}$

$\vec{w}_1 \ldots \vec{w}_n$ $\mathcal{B}$

$[A]_{\mathcal{BA}} = \Sigma$

All matrices admit SVD!

$A = U\Sigma U^* \Rightarrow A^n = U\Sigma^n U^*$

$A = W\Sigma V^* \Rightarrow A^2 = W\Sigma V^* W\Sigma V^*$ $\rightarrow$ it doesn't help with multiplication