$A\vec{x} = \sum_{k=1}^{r} s_k (x, \vec{v}_k)\vec{w}_k$ $\qquad A^*A\vec{v}_k = s_k^2\vec{v}_k \qquad \vec{w}_k = \frac{1}{s_k}A\vec{v}_k$

is this unique? no, can choose basis from multiple eigen values

$A = \sum_{k=1}^{r} s_k(\cdot, \vec{v}_k)\vec{w}_k$

Forms of singular value decomposition

$A = \sum_{k=1}^{r} s_k \vec{w}_k \vec{v}_k^* \qquad A\vec{x} = \sum_{k=1}^{r} s_k \vec{w}_k \underbrace{\vec{v}_k^* \vec{x}}_{\text{innerproduct}} = \sum_{k=1}^{r} s_k w_k (x, \vec{v}_k)$

$W = [\vec{w}_1, \vec{w}_2, \dots, \vec{w}_r]$ - orthonormal system in $\mathbb{R}^n$ ($\mathbb{C}^n$)

$V = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r]$ - orthonormal system in $\mathbb{R}^m$ ($\mathbb{C}^m$)

$A = W \Sigma V^*$ $\qquad$ V, W are isometries (not unitary because not necessarily square)

$\Sigma = \operatorname{diag}(s_1, s_2, \dots, s_r)$

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complete ONS $\vec{v}_1 \dots \vec{v}_r$ & $\vec{w}_1 \dots \vec{w}_r$ to ON Bases in $\mathbb{R}^n$ & $\mathbb{R}^m$ resp.

find ker W^*, ker V^* and apply Gram-Schmidt

$\vec{v}_1 \dots \vec{v}_r\, \vec{v}_{r+1} \dots \vec{v}_n$ ONB in $\mathbb{R}^n$ $\qquad \vec{w}_1, \vec{w}_2 \dots \vec{w}_r, \vec{w}_{r+1} \dots \vec{w}_m$ ONB in $\mathbb{R}^m$

$\tilde{W} = [\vec{w}_1 \dots \vec{w}_m] \qquad \tilde{V} = [v_1 \dots v_n]$ ($\tilde{W}$ and $\tilde{V}$ are unitary)

$\underbrace{A}_{m\times n} = \underbrace{\tilde{W}}_{m\times m} \underbrace{\Sigma}_{r\times r} \underbrace{\tilde{V}^*}_{n\times n}$ $\qquad$ Σ needs to be $m \times n$

these cannot multiply so, we make $\tilde{\Sigma}$

$$\tilde{\Sigma} = \left(\begin{array}{c|c} \begin{matrix} s_1 & & \\ & \ddots & \\ & & s_r \end{matrix} & 0 \\ \hline 0 & \end{array}\right) = \left(\begin{array}{c|c} \Sigma & 0 \\ \hline 0 & \end{array}\right)$$

$A = \tilde{W}\tilde{\Sigma}\tilde{V}^*$

Ex: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = A \quad A^*A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} = \begin{vmatrix} 3-\lambda & 3 \\ 3 & 3-\lambda \end{vmatrix} = 0$

$(3-\lambda)^2 - 9 = 0 \qquad \lambda^2 - 6\lambda + \cancel{9} - 9 = 0$

$\lambda(\lambda - 6) = 0$

$\lambda = 0$ or $\lambda = 6$

$s_1 = \sqrt{6}$

$\vec{v}_1$: $(A^*A - 6I) = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\text{normalized}}$

$w_1 = A\vec{v}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{3}}\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$A = \sqrt{6} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}(1 \;\; 1) = \underbrace{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{W} \underbrace{\sqrt{6}}_{\Sigma} \underbrace{\frac{1}{\sqrt{2}}(1 \;\; 1)}_{V^*}$

complete $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to ONS add $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ +1 \end{pmatrix}$

complete $\frac{1}{\sqrt{3}}$ to ONS $\quad (1 \;\; 1 \;\; 1) \quad x_1 = -x_2 - x_3$

$\vec{x} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{w_2} + x_3 \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{w_3}$

$\vec{w}_2 = \vec{x}_2$

$\vec{w}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{(x_2, w_2)}{\|w_2\|^2} w_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} \underbrace{\frac{1}{\sqrt{6}}\begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}}_{\text{normalized}}$

$A = \tilde{W} \hat{\Sigma} \hat{V}^* =$

A^* - just adjoint of A's decomposition

$$A = \begin{pmatrix} \frac{1}{\sqrt{3}} & -1/\sqrt{2} & -1/\sqrt{6} \\ \frac{1}{\sqrt{3}} & 0 & -1/\sqrt{6} \\ \frac{1}{\sqrt{3}} & 1/\sqrt{2} & 2/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$\max \|A\vec{x}\| \quad \|\vec{x}\| \leq 1$

A - $\text{diag}(s_1, s_2, \ldots s_r)$ $\quad s_1 \geq s_2 \geq s_3 \cdots \geq s_r$

$\underline{\max = s_1}$ largest singular value

$\begin{pmatrix} \begin{matrix} s_1 & & \\ & \ddots & \\ & & s_r \end{matrix} & 0 \\ 0 & \end{pmatrix}$ max is still s_1

Maximum is attained at $\vec{V_1}$ $\quad A^*A\vec{V_1} = s_1^2 \vec{V_1}$