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Norm and condition # of an Operator

Frobenius Norm $\|A\|_F = \sqrt{tr(A^*A)} = \left(\sum |A_{jk}|^2\right)^{1/2}$

Def: Operator norm of A:

$\|A\| = \max_{\|\vec{x}\| \le 1} \|A\vec{x}\| = s_1 \leftarrow$ largest singular value

$= \max_{\|\vec{x}\|=1} \|A\vec{x}\| = \max_{\vec{x} \ne \vec{0}} \frac{\|A\vec{x}\|}{\|\vec{x}\|}$

$\boxed{① \ \|A\vec{x}\| \le \|A\| \cdot \|\vec{x}\|}$

Compare $\|A\|$ and $\|A\|_F$ eigenvalues of A^*A

$\|A\|_F^2 = tr(A^*A) = \sum \lambda_k = \sum s_k^2$ singular values of A

$\|A\|^2 = s_1^2 \leftarrow$ max singular value

$\Rightarrow \|A\| \le \|A\|_F$

Image of the Unit Ball $B = \{\vec{x} : \|\vec{x}\| \le 1\}$

Describe $A(B)$

simple case $A = \Sigma = \begin{pmatrix} s_1 & & & 0 \\ & s_2 & & \\ & & \ddots & \\ 0 & & & s_n \end{pmatrix}$

$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = A\vec{x}$ where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ $\|\vec{x}\|^2 \le 1 \iff \sum |x_k|^2 \le 1$

$\Rightarrow y_k = s_k x_k \qquad x_k = \frac{y_k}{s_k}$

$\Rightarrow \sum |x_k|^2 \le 1 \Rightarrow \sum \left|\frac{y_k}{s_k}\right|^2 \le 1$

$\vec{y}$ that satisfy

$\mathbb{R}^2 \rightarrow$ ellipse

$\mathbb{R}^3 \rightarrow$ ellipsoid

$\mathbb{R}^n \rightarrow$ n-dimensional ellipsoid

$s_k \to$ half axes

Now consider $A = W\Sigma V^*$
Unitary operators do not change norm simply ~~changes basis~~ rotates. Largest axis is $\vec{w}_1$, next is $\vec{w}_2$, etc. If some $s_k = 0$ then the dimension of the ellipsoid will be less than the original unit ball
$A(B)$ - ellipsoid in Ran A

<u>Condition Number of A:</u>

$$\begin{cases} x_1 + x_2 = 2 \\ x_1 + 1.000001 x_2 = 2.000001 + \boxed{\delta} \end{cases}$$

Solution: $x_1 = 1$ $x_2 = 1$ now add error δ

now solution: $x_1 = 1 + \Delta_1$ $x_2 = 1 + \Delta_2$

$$\begin{cases} \Delta_1 + \Delta_2 = 0 \\ \Delta_1 + 1.000001 \Delta_2 = \delta \end{cases}$$

Solution: $\Delta_2 = 1{,}000{,}000\,\delta$ $\Delta_1 = -1{,}000{,}000\,\delta$

$\Rightarrow$ error has magnified million times

$A\vec{x} = \vec{b}$ A is invert $\Rightarrow \vec{x} = A^{-1}\vec{b}$

$A\vec{y} = \vec{b} + \Delta\vec{b}$ small error $\Rightarrow$ $\vec{y} = A^{-1}(\vec{b} + \Delta\vec{b}) = \vec{x} + \Delta\vec{x}$

want to compare error:

$\dfrac{\|\Delta\vec{x}\|}{\|\vec{x}\|}$ to $\dfrac{\|\Delta\vec{b}\|}{\|\vec{b}\|}$ just need relative errors

$\vec{\Delta x} = A^{-1}\vec{\Delta b}$

$$\frac{\|\Delta\vec{x}\|}{\|\vec{x}\|} = \frac{\|A^{-1}\Delta\vec{b}\|}{\|\vec{x}\|} \le \frac{\|A^{-1}\|\cdot\|\Delta\vec{b}\|}{\|\vec{x}\|}$$

use $A\vec{x} = \vec{b}$ $\|\vec{b}\| = \|A\vec{x}\| \le \|A\|\cdot\|\vec{x}\| \Rightarrow \|\vec{x}\| \ge \boxed{\frac{\|\vec{b}\|}{\|A\|}}$

$$\frac{\|A^{-1}\|\cdot\|\Delta\vec{b}\|}{\|\vec{x}\|} \le \frac{\|A\|\|A^{-1}\|\cdot\|\Delta\vec{b}\|}{\|\vec{b}\|}$$

$$\Rightarrow \frac{\|\Delta\vec{x}\|}{\|\vec{x}\|} \le \|A\|\cdot\|A^{-1}\|\,\frac{\|\Delta\vec{b}\|}{\|\vec{b}\|}$$

$\boxed{\|A\|\cdot\|A^{-1}\|}$ → called the condition number of the Matrix A

~~condition number~~

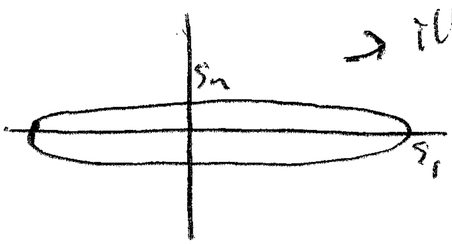
let singular values of A: $s_1 \ge s_2 \ge \cdots \ge s_n$

$\|A\| = s_1$ $\|A^{-1}\| = \frac{1}{s_n}$ $\|A\|\cdot\|A^{-1}\| = \frac{s_1}{s_n}$

when condition # is larger "ill condition"

" " " " small "well condition"

Geometric view



→ ill condition, almost flat, error in s_n is bad, ~~in~~ δs_n in s_1 does not make much difference