Isometries, Unitary, & Orthogonal Matrices. Unitary equivalent operators

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Def $U: X \to Y$

U is isometry if $\|U\vec{x}\| = \|\vec{x}\| \quad \forall \vec{x}$ (Preserves norm)

Prop U is isometry iff $(U\vec{x}, U\vec{y}) = (\vec{x}, \vec{y})$ (Preserves inner product)

Pf ① ($\Rightarrow$) $(\vec{x},\vec{y}) = \frac{1}{4}\sum_{\alpha=\pm1,\pm i} \alpha\|\vec{x}+\alpha\vec{y}\|^2$ (Polarization identity)

$$= \frac{1}{4}\sum_{\alpha=\pm1\pm i} \alpha\|U(\vec{x}+\alpha\vec{y})\|^2 = \frac{1}{4}\sum_{\alpha=\pm1\pm i}\alpha\|\underbrace{U\vec{x}}\, + \alpha \underbrace{U\vec{y}}\|^2$$

Apply polariz. ident. again

$$= (U\vec{x}, U\vec{y})$$

$\Leftarrow$ ② (Trivial)

$$\|U\vec{x}\|^2 = (U\vec{x}, U\vec{x}) = (\vec{x},\vec{x}) = \|\vec{x}\|^2$$

Cor U is isometry iff $\boxed{U^*U = I}$

Pf $(U^*U\vec{x},\vec{y}) = (U\vec{x}, U\vec{y}) = (\vec{x},\vec{y}) = (I\vec{x},\vec{y})$

$\Rightarrow U^*U = I$

(We had a lemma: $(A\vec{x},\vec{y}) = 0 \ \forall \vec{x},\vec{y} \Rightarrow A = \vec{0}$

Cor $(A\vec{x},\vec{y}) = (B\vec{x},\vec{y}) \ \forall \vec{x},\vec{y} \Rightarrow A = B$)

Def $U: X \to Y$ is called unitary if it is isometry & invertible.

Examples: $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ $\mathbb{C}^1 \to \mathbb{C}^2$, $\mathbb{R}^1 \to \mathbb{R}^2$ This is isometry, but NOT invertible.

Prop U is unitary iff $\boxed{U^{-1} = U^*}$

Pf U isom. $\Rightarrow U^*U = I \Rightarrow U^*$ - left inverse for U.

Prop Let $\vec{v}_1 \dots \vec{v}_r$ OS (ONS)
Let U be isometry
Then $U\vec{v}_1 \dots U\vec{v}_r$ OS (ONS)

Preserves orthogonality

Prop Let $\vec{e}_1 \dots \vec{e}_n$ ONB
Then U is unitary iff
$U\vec{e}_1 \dots U\vec{e}_n$ ONB

Examples of Unitary

1. Identity
2. Rotation: $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

because orthonormal basis
$= U\vec{e}_1 \quad = U\vec{e}_2$

or else
$\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$

These are general cases of unitary operators in $\mathbb{R}^2$!

Prop $U: X \to Y$ isometry
U is unitary iff $\dim X = \dim Y$

(Any square isometry is unitary)

Def Orthogonal matrix - unitary matrix with real entries

Def $A: X \to X$, $B: Y \to Y$. A is unitary equivalent to B if $\exists$ unitary $U: X \to Y$ s.t. $B = UAU^*$
$\Rightarrow B = UAU^{-1}$

So A and B are **different** representations of the same object.

This is like similarity in vector spaces. But unitary equivalence applies to inner product spaces.

<u>True/False</u>

U unitary $\Rightarrow U^*$ unitary — T

U_1, U_2 unitary $\Rightarrow U_1 U_2$ unitary — T

U_1, U_2 isometries $\Rightarrow U_1 U_2$ isom. — T

Any isom. is left-invertible.

U isometry $\Rightarrow U^*$ isom. — F
(only if U is unitary)

<u>Example:</u>

$A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ $B \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

unitary equiv?
similar ✓
but not unit. equiv.

<u>Explanation:</u>

① Cols of B are orthogonal
Cols of A aren't

② $\sum_{j,k} |A_{jk}|^2 = \sum_{jk} |B_{jk}|^2$ if $A \cong B$
(unitary equiv)