Bilinear and Quadratic Forms

Bilinear form on $\mathbb{R}^n$: $L(\vec{x},\vec{y})$

1) $L(\alpha\vec{x}_1+\beta\vec{x}_2, \vec{y}) = \alpha L(\vec{x}_1,\vec{y}) + \beta L(\vec{x}_2,\vec{y})$

2) $L(\vec{x}, \alpha\vec{y}_1+\beta\vec{y}_2) = \alpha L(\vec{x},\vec{y}_1) + \beta L(\vec{x},\vec{y}_2)$

$L(\vec{x},\vec{y}) = (A\vec{x},\vec{y})$

$L(\vec{x},\vec{y}) = x_1y_1 + 2x_1y_2 + 3x_2y_1 + 4x_2y_2$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

Quadratic form

1. $Q[\vec{x}] = L(\vec{x},\vec{x})$
2. $Q[\vec{x}]$ is a homogeneous polynomial of 2nd degree of n var. $x_1, x_2 \dots x_n$

$x_k^2 \quad x_k x_j$

Ex: $Q[\vec{x}] = x_1^2 + 4x_1x_2 + 2x_2^2$ $\quad \mathbb{R}^2$

$Q[\vec{x}] = (A\vec{x},\vec{x})$ $\quad A = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix}$

$= (B\vec{x},\vec{x})$

$A\vec{x}, \vec{x}$ $\quad$ Pick $A^t = A$ unique

For any Q: $\exists!\ A = A^t$ s.t. $Q[\vec{x}] = (A\vec{x},\vec{x})$

Ex: of x: $Q[x] = 1$ $\}$ ≤ 1

If A - diagonal then $\vec{x}: (A\vec{x}, \vec{x}) = 1$ is easy to understand.

Diagonalization of Q

$Q[\vec{x}] = (A\vec{x}, \vec{x})$

$\vec{y} = S^{-1}\vec{x} \qquad \vec{x} = S\vec{y}$

$(A\vec{x}, \vec{x}) = (AS\vec{y}, S\vec{y})$

$= (S^{*}AS\vec{y}, \vec{y})$

$A \sim S^{*}AS$

Find S s.t. $S^{*}AS$ diagonalizable.

$A = SDS^{-1} \qquad D = S^{-1}AS$

If $S = U$ - unitary, then $U^{*} = U^{-1}$

Know If $A = A^{*}$ $\exists$ U - unitary, D - diagonal s.t. $A = UDU^{*}$

$A = UDU^{*} \qquad D = U^{*}AU$

Non-orthogonal Diagonalization

$2x_1^2 + 8x_1x_2 + x_2^2$

$2(x_1 + 2x_2)^2 = 2[x_1^2 + 4x_1x_2 + 4x_2^2] = 2x_1^2 + 8x_1x_2 + 8x_2^2$

$= 2(x_1 + 2x_2)^2 - 7x_2^2$

$x_1 + 2x_2 = y_1 \quad x_2 = y_2$

$= 2y_1^2 - 7y_2^2$