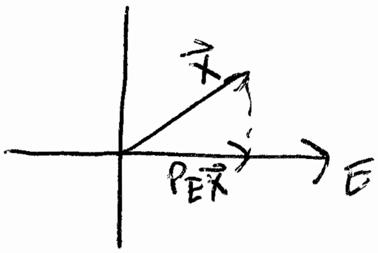
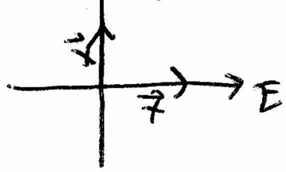
Stefan Janiszewski Notes 4/5/4 ①

HW 3/24 #4

the projection matrix P_E:



$P_E\vec{x} = \lambda\vec{x}$ only true if:
$\vec{x}$ is $\parallel$ or $\perp$ ie eigenvalues 1, 0



Formally: $P_E\vec{x} = \vec{x}_1$ where $\vec{x} = \vec{x}_1 + \vec{x}_2$ $\quad \vec{x}_1 \in E \quad \vec{x}_2 \perp E$

Solve: $P_E\vec{x} = \lambda\vec{x} \quad \vec{x}_1 = \lambda(\vec{x}_1 + \vec{x}_2) \quad \vec{x}_1 = \lambda\vec{x}_1 + \lambda x_2$

$(\lambda - 1)\vec{x}_1 + \lambda\vec{x}_2 = \vec{0}$

are orthogonal i.e. LI $\Rightarrow (\lambda-1)\vec{x}_1 = \vec{0}$ and $\lambda\vec{x}_2 = \vec{0}$

$\Rightarrow \lambda = 1 \quad \vec{x} = \vec{x}_1 \quad \vec{x}_2 = \vec{0} \qquad \lambda = 0 \quad \vec{x} = \vec{x}_2 \quad \vec{x}_1 = \vec{0}$

For $\lambda = 1$ $\operatorname{Ker}(P_E - I) = E$ geo. mult $= r = \dim E$

For: $\lambda = 0$ $\operatorname{Ker}(P_E - 0I) = E^\perp$ geo. mult $= n - r = \dim V - \dim E$

matrix is diagonalizable

$\|\vec{v} - P_E\vec{v}\|^2 = \|\vec{v}\|^2 - \|P_E\vec{v}\|^2$

Notes|

<u>Least Square Solution:</u> (Regression)

$A\vec{x} = \vec{b}$ has a solution iff $\vec{b} \in \operatorname{Ran} A$

if $\vec{b} \notin \operatorname{Ran} A$, but still want solution to eq.:

$A\vec{x} - \vec{b} \rightarrow$ the error, must be minimized i.e. min. norm

cannot be solved but $\|A\vec{x} - \vec{b}\|$ will minimize the error. Can use calculus and partial derivatives but complex. The geometric way:

$\min\|A\vec{x}-\vec{b}\|$ = distance from $\vec{b}$ to $\operatorname{Ran}A$ = $\operatorname{dist}(\vec{b}, \operatorname{Ran}A)$ = $\|\vec{b}-P_{\operatorname{Ran}A}\vec{b}\|$

Solve: $A\vec{x} = P_{\operatorname{Ran}A}\vec{b} \iff A\vec{x}-\vec{b} \perp$ to $\operatorname{Ran}A$ (all collumns of A $\vec{a}_1, \vec{a}_2 \dots \vec{a}_n$)

$\iff (A\vec{x}-\vec{b}, \vec{a}_k) = 0 \quad \forall k$

$\iff \vec{a}_k^*(A\vec{x}-\vec{b}) = 0 \quad \forall k$

$\iff A^*(A\vec{x}-\vec{b}) = 0 \Rightarrow A^*A\vec{x} - A^*\vec{b} = 0 \Rightarrow \boxed{A^*A\vec{x} = A^*\vec{b}}$

Normal Equation

Application:

To find $P_{\operatorname{Ran}A}\vec{b}$ solve $A^*A\vec{x} = A^*b$, mult by A.

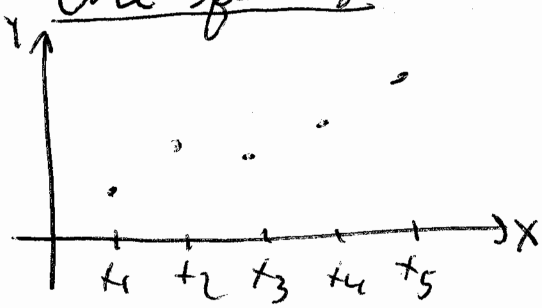
if A^*A is invertible:

$A^*A\vec{x} = A^*\vec{b} \Rightarrow \vec{x} = (A^*A)^{-1}A^*\vec{b}$

$P_{\operatorname{Ran}A}\vec{b} = A\vec{x} = A(A^*A)^{-1}A^*\vec{b}$

$$\boxed{P_{\operatorname{Ran}A} = A(A^*A)^{-1}A^*}$$

<u>line fitting:</u> have some data:



should have $y = ax+b$: Find a and b s.t. $\sum_{k=1}^{n}(ax_k+b-y_k)^2 \to \min$

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \\ \vdots & \\ x_n & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

solve by least square