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Adjoint Operators, Fundamental Subspaces

$A^*A\vec{x} = A^*\vec{b}$: least square solution.

When A^*A is invertible, $\vec{x} = (A^*A)^{-1}A^*\vec{b}$

when is (A^*A) invertible?

$(A^*A)^{-1}$ exists iff columns of A are L I.

Recall: $A^* = \overline{A^T}$ (conjugate of A^T)

properties: 1. $(A+B)^* = A^* + B^*$

2. $(\alpha A)^* = \bar{\alpha} A^*$

3. $(AB)^* = B^*A^*$

4. $(A^*)^* = A$

$\boxed{5.\ (A\vec{x}, \vec{y}) = (\vec{x}, A^*\vec{y})}$

Proof: If A is $m \times n$ matrix, $\vec{x} \in \mathbb{C}^n$, $y \in \mathbb{C}^m$

$$(A\vec{x}, \vec{y}) = \vec{y}^*(A\vec{x}) = (\vec{y}^*A)\vec{x} = (A^*\vec{y})^*\vec{x}$$
$$= (\vec{x}, A^*\vec{y})$$

Adjoint Transformation

$A: V \to W$ (V & W are inner-product spaces)

Define $A^*: W \to V$, such that $(A\vec{v}, \vec{w}) = (\vec{v}, A^*\vec{w})$
$\forall \vec{v} \in V, \forall \vec{w} \in W$

$A = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is orthanormal basis in V

$B = \{\vec{w}_1 \ldots, \vec{w}_m\}$ is orthonormal basis in W

$$[A^*]_{AB} = ([A]_{AB})^*$$

Theorum:

1. $\operatorname{Ker} A = (\operatorname{Ran} A^*)^\perp$
2. $\operatorname{Ker} A^* = (\operatorname{Ran} A)^\perp$
3. $\operatorname{Ran} A = (\operatorname{Ker} A^*)^\perp$
4. $\operatorname{Ran} A^* = (\operatorname{Ker} A)^\perp$

Statements (1) & (2) equivalent: (2) is just (1) applied to A^*.

Same with (3) & (4): (4) is (3) applied to A^*

Statements (4) and (1) are equivalent:

If $\operatorname{Ker} A = (\operatorname{Ran} A^*)^\perp$

then $(\operatorname{Ker} A)^\perp = ((\operatorname{Ran} A^*)^\perp)^\perp = \operatorname{Ran} A^*$

So all 4 statements are equivalent.

Proof of statement 1:

$\vec{x} \in (\operatorname{Ran} A^*)^\perp$

$\Leftrightarrow \vec{x} \perp \operatorname{Ran} A^*$

$\Leftrightarrow (\vec{x}, A^* \vec{y}) = 0 \ \forall \vec{y}$

$\Leftrightarrow (A\vec{x}, \vec{y}) = 0 \ \forall \vec{y}$

$\Leftrightarrow A\vec{x} = \vec{0}$

$\Leftrightarrow \vec{x} \in \operatorname{Ker} A$

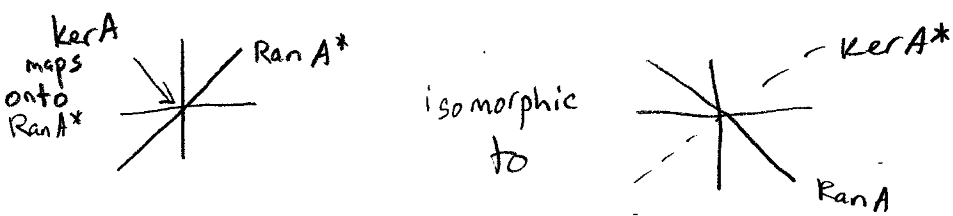
If A is an $m \times n$ matrix,

$A: \mathbb{C}^n \to \mathbb{C}^m$

$\dim \ker A + \dim \operatorname{Ran} A^* = n$

$\dim (\operatorname{Ran} A^*)^\perp + \dim \operatorname{Ran} A^* = n$

Coralary: $\underset{\substack{\shortparallel \\ (\ker A)^\perp}}{\operatorname{Ran} A^*} \to \operatorname{Ran} A$



$(\vec{x}, A\vec{y}) = (A^*\vec{x}, \vec{y})$

$\operatorname{Rank}(A^* A) = \operatorname{Rank} A$

Notes taken by

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