Ilya Medvedev

02/06/2004

$(AB)^T = B^T A^T$ $\qquad (AB)^T_{jk} = (AB)_{kj}$

$= (\text{row } k \text{ of } A)(\text{col } j \text{ of } B)$

$(\text{row } j \text{ of } B^T)(\text{col } k \text{ of } A^T)$

$= (B^T A^T)_{jk}$

Invertible transformation and Invertible Matrices
(AKA Isomorphism)

Special Linear Transform $I_V : V \to V$ $\qquad I\vec{x} = \vec{x}$
identity transformation.

$$I = \underbrace{\begin{pmatrix} 1 & & 0 \\ & 1 & \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{n} \Big\} n \qquad \mathbb{R}^n - \mathbb{R}^n$$

$IA = A \qquad AI = A$

Def: $A : V \to W$ $\qquad$ A is left-invertible if $\exists\, B : W \to V$ s.t. $BA = I$
A is right-invertible if $\exists\, C : W \to V$ s.t. $AB = I$

B - left-inverse of A
C - right-inverse of A

Ex: (i) left-invertable $\qquad \mathbb{R}^1 - \mathbb{R}^2$

$B = (\frac{1}{2}, \frac{1}{2}) \qquad BA = (\frac{1}{2}, \frac{1}{2}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$

$(1,1,1)$ right invertible $\qquad C = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Def: $A: V \to W$

A is invertible if it is both right and left invertible.

---

contrapositive is true of unique not invert

Thm: If $BA = I$ and $AC = I \longrightarrow$

$\to$ 1. $B = C$

2. B and C are unique If $B_1 A = I \to B_1 = B$

$AC_1 = I \to C_1 = C$

Pf: $BAC = B(AC) = BI = B$

$(BA)C = IC = C$

Suppose $B_1 A = I$ $B_1 AC = B_1(AC) = B_1 I = B_1$

$\to B_1 = C$ $C = B$ $B_1 = B$ $\square$

If a matrix is left & right invertible, those matrices are unique and they coincide.

Def: If A invert, it's unique right + left inverse matrix is called inverse A^{-1}.

$A^{-1}A = I$ $AA^{-1} = I$.

Properties of A^{-1}

1. A, B invert $\Rightarrow AB$ invertible

$(AB)^{-1} = B^{-1}A^{-1}$ $B^{-1}A^{-1}AB = B^{-1}IB = I$

$ABB^{-1}A^{-1} = AIA^{-1} = I$

w/ proven

Remark: Only square $(n \times n)$ matrix can be invertable. It sufficient to check only only one identity. $A^{-1}A = I$

$AA^{-1} = I$

Examples of invertable matrices.

Property 2: A invert $\Rightarrow A^T$ is invert & $(A^T)^{-1} = (A^{-1})^T$

1. $I \qquad I^{-1} = I$

2. $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$A^{-1} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

3. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$

$A: V \to W$ is invertible (isomorphism) $\rightarrow$ preserves all properties

then call V & W isomorphic $\rightarrow$ means different representation of the same object

Thm: $A: V \to W$ isomorphism

1. $\vec{v}_1, \vec{v}_2 \dots \vec{v}_n$ is a basis in V

<u>iff</u> $A\vec{w}_1, A\vec{v}_2, \dots, A\vec{v}_n$ is a basis in W.

2. $\vec{v}_1 \dots v_n$ is linearly independent

<u>iff</u> $A\vec{v}_1 \dots A\vec{v}_n$ is LI

3. generating sets...

If $\vec{v}_1 \dots \vec{v}_n$ is a basis $\Rightarrow A\vec{v}_1 \dots A\vec{v}_n$ is a basis

$\vec{w} = \sum \alpha_k A\vec{v}_k$ ___ represent $A^{-1}\vec{w} = \sum \alpha_k \vec{v}_k$

$\vec{v} = \sum \dots$

Equations:

(*) $A\vec{x} = \vec{b}$ A is a linear transform $A: V \to W$

iff $\Leftarrow$ A is invert, then $\vec{b} \in W$

$\vec{x} = A^{-1}\vec{b}$ is a unique solution of (*)

for any b, ~~this~~ there is a unique solution, then A is invertible