2/11/04

Systems of linear eqns

① $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n = b_m \end{cases}$$

m eqns ; n unknowns

Solve: find all n-tuples $(x_1 \dots x_n)$ satisfying all the eqns simultaneously

② Matrix form

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ & \cdots & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$A\vec{x} = \vec{b}$ find all $\vec{x}$ to make the equality true

③ Vector eqn

$\vec{a}_k$ - Column #k of A

$$\vec{a}_k = \begin{pmatrix} a_{k1} \\ a_{k2} \\ \vdots \\ a_{km} \end{pmatrix} \qquad \vec{x}_1\vec{a}_1 + \vec{x}_2\vec{a}_2 + \dots \vec{x}_n\vec{a}_n = \vec{b}$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 3x_1 + 2x_2 + x_3 = 7 \\ 2x_1 + x_2 + 2x_3 = 1 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 7 \\ 2 & 1 & 2 & 1 \end{array}\right)$$

Augmented Matrix of the system

Gauss-Jordan elimination AKA row reduction

- Interchange 2 rows
- multiply a row by a non-zero #
- row replacement ⇒ replace row #k by $r_k + ar_j$

eliminate all non diagonal entries

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1 \\ 3 & 2 & 1 & 7 \\ 2 & 1 & 2 & 1 \end{array}\right) \begin{array}{c} \\ -3r_1 \\ -2r_1 \end{array} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & 4 \\ 0 & -3 & -4 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & \boxed{1} & 2 & -1 \\ 0 & -3 & -4 & -1 \end{array}\right) {+3r_2} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -4 \end{array}\right)$$

Back Substitution

① $2x_3 = -4$; $x_3 = -2$

② $x_2 + 2x_3 = -1$

$x_2 - 4 = -1$

$x_2 = 3$

③ $x_1 + 2(3) + 3(-2) = 1$

$x_1 = 1$

$$\vec{x} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & \boxed{1} & -2 \end{array}\right) \begin{array}{l} -3r_3 \\ -2r_3 \\ \\ \end{array} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right) \begin{array}{l} -2r_2 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right) !$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 2 \\ 0 & 2 & 3 & 1 & 3 \\ 0 & 0 & 0 & 2 & 2 \end{array}\right)$$

Echelon form

1) All 0 rows are below all non-zero rows
2) The leading entry of a row is strictly to the right of the leading entry of the previous row

Leading entry: leftmost non-zero

$x_4 = 1$ $\quad$ x_3 - free variable → not enough eqns to define it

$x_2 = \frac{1}{2}(3 - 3x_3 - 1) = 1 - \frac{3x_3}{2}$

$x_1 = 2 - (1 - \frac{3}{2}x_3) - 2x_3 = 1 - \frac{x_3}{2}$

$$\vec{x} = \begin{pmatrix} 1 - \frac{1}{2}x_3 \\ 1 - \frac{3}{2}x_3 \\ x_3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix}$$