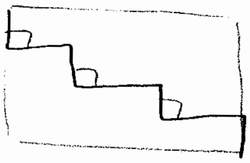
Analyzing the Pivots



Echelon

Reduced Echelon all pivots = 1 and all entries in the pivot column (except pivot) are zero

* Question: when a system is inconsistent (no solution)
- A system is inconsistent iff we get a row $(0\,0\,0\,0\,0\,1\;b)$ $b \neq 0$ in echelon form $0x_1 + 0x_2 + \dots + 0x_n = b$

1. A solution is unique (if $\exists$) iff there is no free variables (iff $\exists$ pivot in every column of A)
2. A solution exists for every right side $\vec{b}$ iff $\exists$ a pivot in every row of A
3. A solution of $A\vec{x} = \vec{b}$ exists and is unique for any $\vec{b}$ iff $\exists$ pivot in every row and column.

$$\vec{v}_1 \dots \vec{v}_n$$
$$x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n = \vec{v} \quad \text{basis-exists + unique}$$
$$A = [\vec{v}_1\ \vec{v}_2\ \dots\ \vec{v}_n]$$

1. A system $\vec{v}_1 \dots \vec{v}_n$ is linearly independent iff $A = [\vec{v}_1\ \vec{v}_2 \dots \vec{v}_n]$ have pivots in every column
2. A system $\vec{v}_1 \dots \vec{v}_n$ is generating iff A has pivots in every row
3. A system $\vec{v}_1 \dots \vec{v}_n$ is a basis iff A has pivots in every row and column.

* Observation - Any row or column can have ≤ 1 pivot

$\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}3\\2\\1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}$ are these four vectors linearly independent?
- no - if joined together and do row operations cannot have more than 3 pivots (3 rows) but there are four columns so not every column will have a pivot

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1. Any linearly independent system $\vec{v}_1 \dots \vec{v}_m \in \mathbb{R}^n$ cannot have more than n vectors in it
   $\#$ of pivots $\leq n$ $= \#$ of rows
   Pivot in every column $\Rightarrow \#$ of column $= \#$ of pivots $\leq n$
2. Any generating (spanning) system in $\mathbb{R}^n$ has at least n vectors in it
3. Any basis in $\mathbb{R}^n$ has exactly n vectors in it

- Corrollary of statement 3: If $\vec{v}_1 \dots \vec{v}_n$ and $\vec{w}_1 \dots \vec{w}_m$ are bases in $V \Rightarrow m = n$

- Prove invertible matrix must be square
  A is invertible $\Rightarrow A$ is square
  - If $BA = I \Rightarrow A\vec{x} = \vec{0}$ is unique
    $BA\vec{x} = B\vec{0}$
    $\vec{x} = \vec{0} \Rightarrow \exists$ pivot in every column
  - If $AC = I$ then $C\vec{b}$ solves $A\vec{x} = \vec{b}$ $\forall \vec{b} \Rightarrow \exists$ pivot in every row
  - $\#$ of columns $= \#$ of rows $\Rightarrow$ square

* <u>Proposition</u>: matrix is invertible iff $\exists$ pivot in every row and coleumn

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

$$-3r_1 \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -5 & -3 & 1 \end{array}\right)^{5} \sim \left(\begin{array}{cc|cc} 5 & 10 & 5 & 0 \\ 0 & -5 & -3 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 5 & 0 & -1 & 2 \\ 0 & -5 & -3 & 1 \end{array}\right)\begin{matrix} 1/5 \\ -1/5 \end{matrix}$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & -1/5 & 2/5 \\ 0 & 1 & 3/5 & -1/5 \end{array}\right)$$

$$A^{-1}$$

$A^{-1}b$ solves $A\vec{x} = \vec{b}$

$A\vec{x} = \vec{e}_1$ solution gives first column A^{-1}
$A\vec{x} = \vec{e}_2$ solves second column
$\vdots$
$A\vec{x} = \vec{e}_n$

(method solves all these equations simultaneously)

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