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Subspaces, Dimensions, Fundamental subspaces of a matrix, structure of general solution of a linear solution

Subspaces

Def - W - vector space $V \subset W$ V is a subspace of W if,

1. $\forall \vec{x}, \vec{y} \in V$ $\vec{x} + \vec{y} \in V$
2. $\forall \vec{x} \in$ $\forall$ scalar α $\alpha \vec{x} = V$

1 & 2 $\Leftrightarrow$ $\forall x, y$ $\forall \alpha, \beta$ scalars $\alpha \vec{x} + \beta \vec{y} \in V$

V is closed under vector addition and scalar multiplication.

Obs - A subspace is a vector space

Ex: 1. Trivial subspaces of W: $\{0\}$, W

2. W, $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n \in W$ subspace $V = \left\{ \begin{array}{l} \text{all possible linear} \\ \text{combinations } \sum_{k=1}^{n} \alpha_k \vec{v}_k \end{array} \right\}$

span $(v_1, v_2 \ldots v_n)$ or $\mathcal{L}(v_1, v_2 \ldots v_n)$

3. Fundamental subspaces of a matrix

A $m \times n$ $\mathbb{R}^n \to \mathbb{R}^m$

1. Range of A, $\text{Ran} A = \{ \vec{y} \in \mathbb{R}^m : \text{s.t. } \vec{y} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R} \}$

Prop. $A\vec{x} = \vec{b}$ has a solution iff $b \in \text{Ran} A$

If $\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n$ are columns of A then $\text{Ran} A = \text{span}(\vec{a}_1, \vec{a}_2 \ldots \vec{a}_n)$

(row by coordinate rule) $\vec{x} = (x_1, x_2, \ldots x_n)^T$

$$A\vec{x} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 + \ldots + x_n \vec{a}_n$$

Column space $\text{Col} A$

2. Null Space of A (Kernel of A)

$\ker A = \text{Nul} A = \{ \vec{x} \in \mathbb{R}^n : A\vec{x} = 0 \}$

3. $\text{Ran} A^T$ (row space of A)

4. $\ker A^T$ (left null space)

$A^T \vec{x} = \vec{0}$ $\quad (A^T \vec{x})^T = \vec{0}^T$ $\quad \vec{x}^T A = \vec{0}^T$

Structure of solution of $A\vec{x} = b$:

$A\vec{x} = \vec{b}$ $\quad \vec{b} \in \operatorname{Ran} A$

$A\vec{x} = \vec{0}$ $\quad$ Homogeneous equation

~~Thm: Gen Sol of $A\vec{x} = \vec{b}$~~

Thm: Gen Sol of $A\vec{x} = b$ $= \underbrace{\begin{bmatrix} \text{Gen Sol.} \\ \text{of } A\vec{x} = 0 \end{bmatrix}}_{\ker A} + \begin{bmatrix} \text{A particular} \\ \text{solution of} \\ A\vec{x} = \vec{b} \end{bmatrix}$

Pf. Let $A\vec{x_p} = \vec{b}$; $A\vec{x_n} = \vec{0}$

Then $A(\vec{x_p} + \vec{x_n}) = A\vec{x_p} + A\vec{x_n}$

$\vec{b} + \vec{0} = \vec{b}$

so $\vec{x_p} + \vec{x_n}$ solves $A\vec{x} = \vec{b}$

Let $A\vec{x_p} = \vec{b}$ and also $A\vec{x} = \vec{b}$

$A(\vec{x} - \vec{x_p}) = A\vec{x} - A\vec{x_p} = \vec{b} - \vec{b} = \vec{0}$

$\underbrace{\vec{x} - \vec{x_p}}_{\vec{x_h}} \in \ker A$

$\vec{x} = \vec{x_p} + \vec{x_h}$

$$\begin{pmatrix} 2 & 3 & 1 & 4 & -9 \\ 1 & 1 & 1 & 1 & -3 \\ 1 & 1 & 1 & 2 & -5 \\ 2 & 2 & 2 & 3 & -8 \end{pmatrix} \vec{x} = \begin{pmatrix} 17 \\ 6 \\ 8 \\ 14 \end{pmatrix} \qquad \vec{x} = \underbrace{\begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}}_{\substack{\text{particular} \\ \text{solution} \\ x_p}} + \underbrace{s\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}}_{\in \text{ kernel } A}$$