Notes 02/18/04

Rank of a matrix (Rank theorem)

The subspaces of $\mathbb{R}^2$:
of dimension 0: $\{\vec{0}\}$ of dimension 1: all lines through origin $\vec{0}$
of dimension 2: $\mathbb{R}^2$

for $\mathbb{R}^3$: $\{\vec{0}\}$, all lines through origin $\vec{0}$, all planes through origin $\vec{0}$, $\mathbb{R}^3$

<u>Def:</u> Rank of A = $\dim(\operatorname{Ran} A) = \dim(\operatorname{Col} A) = \operatorname{Rank} A$

Computing basis in fundamental subspaces

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 2 \\ 1 & 1 & -1 & -1 & 0 \end{pmatrix} \begin{array}{l} \\ -2r_1 \\ -3r_1 \\ -r_1 \end{array} \sim \begin{pmatrix} 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & -3 & -3 & -1 \\ 0 & 0 & -3 & -3 & -1 \\ 0 & 0 & -3 & -3 & -1 \end{pmatrix} \begin{array}{l} \\ \\ -r_2 \\ -r_2 \end{array} \sim \begin{pmatrix} \boxed{1} & 1 & 2 & 2 & 1 \\ 0 & 0 & \boxed{-3} & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

to compute basis in $\operatorname{Col} A$:
Take pivot <u>columns</u> of A (non echelon)

$$\operatorname{Col} A = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \\ -1 \end{pmatrix}$$

The basis in the row ~~space~~: $\operatorname{Col} A^T$:
Take pivot <u>rows</u> of echelon form of A and write vertically

$$\operatorname{Col} A^T = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \\ -3 \\ -1 \end{pmatrix}$$

Reduced Echelon:

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & -3 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \times 3 \\ \\ \\ \\ \end{array} \sim \begin{pmatrix} 3 & 3 & 6 & 6 & 3 \\ 0 & 0 & -3 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} +2r_2 \\ \\ \\ \\ \end{array} \sim \begin{pmatrix} 3 & 3 & 0 & 0 & 1 \\ 0 & 0 & -3 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \div 3 \\ \div(-3) \\ \\ \\ \end{array} \sim \begin{pmatrix} 1 & 1 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

<u>Solving:</u> x_5 = free x_4 = free $x_3 = -x_4 - \frac{x_5}{3}$ x_2 = free
$x_1 = -x_2 - \frac{x_5}{3}$ This is a basis Ker A

$$\begin{pmatrix} -x_2-\frac{x_5}{3} \\ x_2 \\ -x_4-\frac{x_5}{3} \\ x_4 \\ x_5 \end{pmatrix} = x_2\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_5\begin{pmatrix} -1/3 \\ 0 \\ -1/3 \\ 0 \\ 1 \end{pmatrix}$$

these three vectors form a basis in $\operatorname{Ker} A$ ③

to find $\operatorname{Ker} A^T$ you must repeat this process with A^T

<u>Rank Theorem:</u> $\operatorname{Rank} A = \operatorname{Rank} A^T$

if $A = m \times n$

$\operatorname{Rank} A + \dim \operatorname{Ker} A = n \quad \operatorname{Rank} A + \dim \operatorname{Ker} A^T = m$