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Proofs: how to write them?

Mathematical statement in the form: $(A) \Rightarrow (B)$

"If A, then B"

-or- $(A) \Leftrightarrow (B)$ means $(A) \Rightarrow (B)$ and $(B) \Rightarrow (A)$

"(A) holds iff (B) holds"

Proof: • Split $(A) \Rightarrow (B)$ into elementary steps by demonstrating that A has some other implications because of some theorem/property: $(A) \Rightarrow (A_1) \Rightarrow (A_2) \Rightarrow \cdots \Rightarrow (A_n) \Rightarrow (B)$

• In cases $(A) \Leftrightarrow (B)$ one must either use a series of equivalences, or prove $(A) \Rightarrow (B)$ and $(B) \Rightarrow (A)$ separately

Homework Problem #5

$A : X \to Y$

Prove: V subspace of $X \Rightarrow \dim AV \leq \operatorname{rank} A$

Since $V \subset X \Rightarrow AV \subset AX = \operatorname{Ran} A$

$\Rightarrow AV$ is subspace of $\operatorname{Ran} A \underset{(*)}{\Rightarrow} \dim AV \leq \dim \operatorname{Ran} A =: \operatorname{rank} A$

(*) Prop: If $V \subset W$ then $\dim V \leq \dim W$

Pf: Let $\dim V = n$, $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ be a basis in V.
then $\vec{v}_1 \ldots \vec{v}_n$ is LI. Note all $\vec{v}_k \in W$.
So $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ LI in W. $\Rightarrow \dim W \geq n$.
(Any LI system in W cannot have more than $\dim W$ vectors) $\square$

Overview of dimension and rank

dimension: (informally) - how "big" the space -or- how many coordinates we need to describe a vector

Let $\dim V = n$, $\vec{v}_1, \vec{v}_2 \dots \vec{v}_n$ is LI. Then $\vec{v}_1, \vec{v}_2 \dots \vec{v}_n$ is a basis.

Pf: If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is not generating then $\exists \vec{v}_{n+1} \notin \operatorname{span}(\vec{v}_1, \vec{v}_2, \dots \vec{v}_n)$

Then $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{v}_{n+1}$ is LI consists of $n+1 > n$ vectors $\boxed{\Rightarrow \Leftarrow}$

contradiction

-or-

Pf: Moving to $\mathbb{R}^n$

If $V = \mathbb{R}^n$

Form $A = [\vec{v}_1\ \vec{v}_2 \dots \vec{v}_n]$

$\vec{v}_1 \dots \vec{v}_n$ is LI $\Leftrightarrow$ pivots in every column

$\Rightarrow$ n pivots $\Rightarrow$ pivots in every row

$\Rightarrow \vec{v}_1 \dots \vec{v}_n$ generating $\Rightarrow$ basis.

General case:
$\dim V = n$, so $\mathbb{R}^n \cong V$
So
WLOG assume that $V = \mathbb{R}^n$

WLOG - Without Loss of Generality ← Clutch

Rank - rank $= k$, A collapses X to a k-dim space and moves it to Y.

$A: X \to Y$

$\mathbb{R}^n \quad \mathbb{R}^m$

Consider: $A\vec{x} = \vec{0}$ (Alternative point of view)

Ask: How many Linearly Independent equations are there?

= How many LI rows does A have?

$\dim \operatorname{Ran} A^T = \dim \operatorname{Ran} A$

In practice, $\boxed{\text{rank } A = \#\text{ pivots in } A}$

Why does this stuff matter? - Computers do the computation, but you don't want to get FIRED!

Need to know what to expect:
unique solution, infinitely many?
How many free parameters $=$ dim Ker A