Venessa Reese

MA 54 classnotes 2/25/03

Change of Basis

Let V $\{\vec{v}_1, \vec{v}_2, \vec{v}_3 \dots \vec{v}_n\}$ be some basis $\mathcal{B}$

$\forall\ \vec{v} \in V$ admits unique representation

$$\vec{v} = \sum_{k=1}^{n} x_k \vec{v}_k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} =: [\vec{v}]_{\mathcal{B}}$$

- called coordinate representation of $\vec{v}$ in the basis $\mathcal{B}$ or relative to the basis $\mathcal{B}$

Coord representation $\vec{v} \mapsto [\vec{v}]_{\mathcal{B}}$ is an isomorphism between V and $\mathbb{R}^n$

Suppose 2 spaces X and Y and linear transformation $T: X \to Y$

$A = \{\vec{a}_1, \vec{a}_2, \dots \vec{a}_n\} \to$ basis in X

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots \vec{b}_n\} \to$ basis in Y

take $\vec{x} \in X$ and let $\vec{y} = T\vec{x}$

How $[\vec{y}]_{\mathcal{B}}, [\vec{x}]_A$ are related?

a unique

$\exists!$ matrix $[T]_{\mathcal{B}A}$ s.t. $[\vec{y}]_{\mathcal{B}} = [T]_{\mathcal{B}A} [\vec{x}]_A$

How to find $[T]_{\mathcal{B}A}$?

Take $[T\vec{a}_1]_{\mathcal{B}}, [T\vec{a}_2]_{\mathcal{B}} \dots [T\vec{a}_n]_{\mathcal{B}}$ and join them in a matrix.

Matrix of composition

Suppose we have spaces and bases

$$\begin{tikzcd}
X \arrow[r, "T_2"] \arrow[d, "\vec{x} \mapsto [\vec{x}]_A"] & Y \arrow[r, "T_1"] \arrow[d, "\vec{y} \mapsto [\vec{y}]_{\mathcal{B}}"] & Z \arrow[d, "\vec{z} \mapsto [\vec{z}]_{\mathcal{C}}"] \\
\mathbb{R}^n \arrow[r, "{[T_2]_{\mathcal{B}A}}"'] & \mathbb{R}^m \arrow[r, "{[T_1]_{\mathcal{C}\mathcal{B}}}"'] & \mathbb{R}^r
\end{tikzcd}$$

$A \quad \mathcal{B} \quad \mathcal{C}$

where computer does operation!

Let $T = T_1 T_2 : X \to Z$

$[T]_{\mathcal{C}A} = [T_1]_{\mathcal{C}\mathcal{B}} [T_2]_{\mathcal{B}A}$

Vanessa Ree

2/25/04

Change of coordinate matrix

$X \qquad A = \{\vec{a}_1 \dots \vec{a}_n\} \qquad \beta = \{\vec{b}_1 \dots \vec{b}_n\}$

$[I]_{\beta A}$ ← takes coordinates in A basis and gives out coordinates in β basis

$$[\vec{x}]_\beta = \underbrace{[I]_{\beta A}}_{\text{change of coordinate matrix}} [\vec{x}]_A$$

Remark:

$$[I]_{A\beta} = [I]_{\beta A}^{-1}$$

Ex $\quad T = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \qquad \beta = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$

Want to find $[T]_{\beta\beta} \qquad s = \{\vec{e}_1, \vec{e}_2\}$ - standard basis

Know that:

$$[T]_{ss} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$$

$$[T]_{\beta\beta} = [I]_{\beta s} [T]_{ss} [I]_{s\beta}$$

$$[I]_{s\beta} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad [I]_{\beta s} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$[T]_{\beta\beta} = \frac{1}{3}\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

if $[I]_{s\beta} = Q$ then $[I]_{\beta s} = Q^{-1}$

Ex

$\mathbb{P}_1 \qquad A = \{1, 1+t\} \qquad \beta = \{1+2, 1-2t\} \qquad s = \{1, t\}$

Find: $[I]_{\beta A} = [I]_{\beta s}[I]_{sA}$

$$[I]_{sA} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$[I]_{\beta s} = [I]_{s\beta}^{-1} \qquad [I]_{s\beta} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$$

$$[I]_{\beta s} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}^{-1}$$

$$[I]_{\beta A} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$