Determinants

Determinant - signed volume (n-dimensional)

1 dim volume: length

2 dim volume: area

3 dim volume: volume

Parallelepiped: 2d 

3d 

n vectors in $\mathbb{R}^n$: $\vec{v}_1 \dots \vec{v}_n$

Parallelepiped:

$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots \alpha_n \vec{v}_n \qquad 0 \le \alpha_k \le 1$

$D(\vec{v}_1, \dots \vec{v}_n)$ — "volume" of parallelepiped

Volume = base × height

$D(\alpha \vec{v}_1, \vec{v}_2 \dots \vec{v}_n) = \alpha D(\vec{v}_1, \vec{v}_2, \dots \vec{v}_n)$

Properties:

1. a) $D(\vec{v}_1, \dots \alpha \vec{v}_k \dots \vec{v}_n) = \alpha D(\vec{v}_1 \dots \vec{v}_k \dots \vec{v}_n)$

   b) $D(\vec{v}_1, \dots, \vec{u}_k + \vec{v}_k, \dots \vec{v}_n) = D(\vec{v}_1 \dots \vec{u}_k \dots \vec{v}_n) + D(\vec{v}_1 \dots \vec{v}_k \dots \vec{v}_n)$

   (linearity)

2. $D(\vec{v}_1, \dots \underbrace{\vec{v}_k + \alpha \vec{v}_j}_{k}, \dots \vec{v}_n) = D(\vec{v}_1 \dots \vec{v}_k \dots \vec{v}_n)$

   Preservation of column replacement

3. Antisymmetry: $D(\vec{v}_1, \dots \vec{v}_j \dots \vec{v}_k \dots \vec{v}_n) = -D(\vec{v}_1 \dots \vec{v}_k \dots \vec{v}_j \dots \vec{v}_n)$

Proof: $D(\dots \vec{v}_j \dots \vec{v}_k \dots) = D(\dots \vec{v}_j \dots \vec{v}_j + \vec{v}_k \dots) = D(\dots \vec{v}_j - (\vec{v}_j + \vec{v}_k) \dots \vec{v}_j + \vec{v}_k \dots)$

$= D(\dots -\vec{v}_k \dots \vec{v}_j \dots) = -D(\dots \vec{v}_k \dots \vec{v}_j \dots)$

4. Normalization: $D(\vec{e}_1, \dots \vec{e}_n) = 1$

A $n \times n$ $\quad \det(A)$

$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \det\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right)$

$\det I = 1$

5. If A has column of 0's $\to \det A = 0$

$D(\vec{v}_1, \dots \vec{0} \dots \vec{v}_n) = D(\vec{v}_1, \dots a\vec{0} \dots \vec{v}_n)$

$\det(A) = a \det(A) \quad \forall a \to \det A = 0$

6. A has 2 equal columns $\to \det(A) = 0$

$\det A = -\det A \to \det A = 0$

7. If one col. of A is mult. of another $\to \det A = 0$

8. If one col is linear comb. of others $\to \det A = 0$

(or A not invertible $\to \det A = 0$)

9. $\det \begin{pmatrix} a_1 & & 0 \\ & a_2 \ddots & \\ 0 & & a_n \end{pmatrix} = a_1 a_2 \cdots a_n$

10. $\det \begin{pmatrix} a_1 & & * \\ & a_2 \ddots & \\ 0 & & a_n \end{pmatrix} = a_1 \cdots a_n = \det \begin{pmatrix} a_1 & & 0 \\ & a_2 \ddots & \\ * & & a_n \end{pmatrix}$

Computing determinants

Ex: $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 3 \end{vmatrix} \overset{-C_1}{=} -\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = -2$

11. $\det A^T = \det A$

12. $\det(AB) = \det A \cdot \det B$