Review

$\mathbb{R}^n$ $\mathbb{C}^n$ $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \\ 4 \end{pmatrix}$

Transpose - interchange rows and columns.

$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T$

Bases, Linear Independence, generating sets

V $(\mathbb{R}^{10}, \mathbb{R}^{120})$ $\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n$

Def. Linear Combination $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \ldots \alpha_n \vec{v}_n$

$$2\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ 13 \\ 19 \end{pmatrix}$$

Def. A system of vectors is called a basis if any $\vec{v} \in V$ has unique representation.

$$\vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \ldots + \alpha_n \vec{v}_n = \sum_{k=1}^{n} \alpha_k \vec{v}_k$$

Examples: 1. $V = \mathbb{R}^n$ $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots \vec{e}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \#k, \ldots \vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \ldots + x_n \vec{e}_n$ $\vec{e}_1 \ldots \vec{e}_n$ - Standard Basis in $\mathbb{R}^n$

2. P_n $\vec{e}_0 = 1$ $\quad p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$

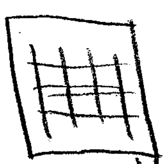
$\vec{e}_1 = t$ $\quad p = a_0 \vec{e}_0 + \vec{a}_1 \vec{e}_1 + \ldots + \vec{a}_n \vec{e}_n$

$\vec{e}_2 = t^2$

$\vdots$

$\vec{e}_n = t^n$

$M_{m \times n}$



$m \cdot n$ entries

α_k - coordinates

A system $\vec{v}_1 \ldots \vec{v}_n$ is a basis if $\forall \vec{v}$ equation $x_1 \vec{v}_1 + x_2 \vec{v}_2 + \ldots + x_n \vec{v}_n = \vec{v}$ has a unique solution.

2 Statements: exists, unique

Existence: Def. A system $\vec{v}_1 \ldots \vec{v}_n$ is called a generating (or spanning, or complete) system if any $\vec{v} \in V$ admits a representation.

$\vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \ldots + \alpha_n \vec{v}_n$ $\quad \forall \vec{v}$ equ. $x_1 \vec{v}_1 + x_2 \vec{v}_2 + \ldots + x_n \vec{v}_n = \vec{v}$ has a solution.

Any basis is a generating set and LI.

Uniqueness Def. $\sum_{k=1}^{n} \alpha_k \vec{v}_k$ is called trivial if all $\alpha_k = 0$.

Def. $\vec{v}_1 \ldots \vec{v}_n$ is called linear independent (LI) if $\vec{0}$ can be represented only as trivial linear comb. of $\vec{v}_1 \ldots \vec{v}_n$.

$x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n = \vec{0}$ has only trivial solution (all $x_k = 0$)

Theorem Let $\vec{v}_1 \ldots \vec{v}_n$ be generating and LI. Then $\vec{v}_1 \ldots \vec{v}_n$ is a basis.

Pf $\vec{v}_1 \ldots \vec{v}_n$ is generating. Take any $\vec{v}$. $\vec{v} = \sum_{k=1}^{n} \alpha_k \vec{v}_k$

Let $\vec{v} = \sum \tilde{\alpha}_k \vec{v}_k$ $\sum_{k=1}^{n} (\alpha_k - \tilde{\alpha}_k)\vec{v}_k = \sum_{k=1}^{n} \alpha_k \vec{v}_k - \sum_{k=1}^{n} \tilde{\alpha}_k \vec{v}_k = \vec{v} - \vec{v} = \vec{0}$

Since $\vec{v}_1 \ldots \vec{v}_n$ is LI then $\alpha_k - \tilde{\alpha}_k = 0$ $\forall k$. So $\alpha_k = \tilde{\alpha}_k$ and representation is unique.

Linear dependence.

Prop. 2.8