Justification for row expansion cofactor formula for the inverse

13 March

Matrix A without row 1, column 1 $= A_{11}$

$$\left[\begin{array}{c|c} a_{11} & * \\ \hline * & A_{11} \end{array}\right]$$

1. $$\left|\begin{array}{c|ccc} a_{11} & 0 & 0 \cdots & 0 \\ \hline * & & A_{11} & \end{array}\right| = a_{11} \det A_{11}$$

2. $$\left|\begin{array}{c|c|ccc} 0 & a_{12} & 0 & 0 \cdots & 0 \\ \hline & * & & & \end{array}\right| = -\left|\begin{array}{c|ccc} a_{12} & 0 & 0 \cdots & 0 \\ \hline * & & A_{12} & \end{array}\right| = -a_{12} \det A_{12}$$

3. $$\left|\begin{array}{cc|c|ccc} 0 & 0 & a_{13} & 0 & \cdots & 0 \\ \hline & & * & & & \end{array}\right| = -\left|\begin{array}{c|c|ccc} 0 & a_{13} & 0 & \cdots & 0 \\ \hline & * & & & \end{array}\right| = +\left|\begin{array}{c|ccc} a_{13} & 0 & \cdots & 0 \\ \hline * & & A_{13} & \end{array}\right| = a_{13} \det A_{13}$$

(negatives cancel)

If we switch 1st + 3rd column, the order of the 1st and 2nd columns in A_{13} is switched.

k: $$\left|\begin{array}{ccc|c|ccc} 0 & \cdots & 0 & a_{1k} & 0 & \cdots & 0 \\ \hline & & & * & & & \end{array}\right| = (-1)^{(1+k)} a_{1k} \det A_{1k}$$

$$\left|\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline & A_1 & & \end{array}\right| = \left|\begin{array}{cccc} a_{11} & 0 & \cdots & 0 \\ \hline & A_1 & & \end{array}\right| + \left|\begin{array}{cccc} 0 & a_{12} & 0 \cdots & 0 \\ \hline & A_1 & & \end{array}\right| + \cdots + \left|\begin{array}{cccc} 0 & 0 & \cdots & a_{1n} \\ \hline & A_1 & & \end{array}\right|$$

(Linearity: $D(\vec{u}_1 + \vec{v}_1, \vec{v}_2 \ldots \vec{v}_n) = D(\vec{u}_1, \ldots \vec{v}_n) + D(\vec{v}_1 + \ldots \vec{v}_n)$)

$= \Sigma (-1)^{1+k} (a_{1k}) \det A_{1k}$

To expand in second row, switch rows 1 & 2, expand with new first row.

Get $-\Sigma(-1)^{k+1}(a_{2k}) \det(A_{2k})$

$= \Sigma(-1)^{2+k}(a_{2k}) \det(A_{2k})$

and so on...

Definition: $C_{jk} = (-1)^{j+k} \det A_{jk}$ $\qquad$ C_{jk} – cofactor

$$\det A = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} \det A_{jk}$$
$$= \sum_{k=1}^{n} a_{jk} C_{jk}$$

Cofactor matrix $=: \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{2n} & & C_{nn} \end{pmatrix}$

Theorum: Let A be invertible
Then $A^{-1} = (1/\det A)\, C^T$

eg: $A = \begin{pmatrix} 1 & 2 \\ 5 & 7 \end{pmatrix} = \dfrac{1}{7-10} \begin{pmatrix} 7 & -5 \\ -2 & 1 \end{pmatrix}^T = \dfrac{-1}{3} \begin{pmatrix} 7 & -2 \\ -5 & 1 \end{pmatrix}$

Generally, this is not conveniant: takes at least $n!$ multiplications to compute det by cofactors

Proof of Theorem: $(A \cdot C^T)_{jj} = \sum_{k=1}^{n} A_{jk} \cdot C_{jk}$ $\qquad$ (C_{jk} = row # k of C, column # k of C^T)

$= \det A$

$(A \cdot C^T)_{jk} = \sum_{\ell=1}^{n} A_{j\ell} \cdot C_{k\ell}$ $\qquad$ $\Big|$ $\sum_{\ell=1}^{n} a_{k\ell} \cdot C_{k\ell} = \det A$

$= \begin{vmatrix} A \\ r_j \\ r_j \end{vmatrix} k$ (row # k replaced by row # j)

$= 0$ as $\det A = 0$ when A has 2 equal rows.

$A \cdot C^T = (\det A)\, I$