MA54. 3.5. Fri. #1.

Formal definition of determinant.
Permutations. Relations with Combinatories.

1. determinant was not well defined.

2. getting formal det

$\det A$, $\vec{a}_1, \vec{a}_2 \cdots$: columns of A
M(n×n)

$$\vec{a}_1 = \sum_{i=1}^{n} a_{i1} \vec{e}_i$$

$$\begin{aligned}\det A &= D(\vec{a}_1, \vec{a}_2 \cdots \vec{a}_n) \\ &= D(\sum_i a_{i1} \vec{e}_i, \vec{a}_2, \cdots \vec{a}_n) \\ &= \sum_{i=1}^{n} a_{i1} D(\vec{e}_i, \vec{a}_2, \cdots \vec{a}_n) \\ &\vdots \\ &= \sum_{i_1=1} \sum_{i_2=1} \cdots \sum_{i_n=1} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} \underline{D(\vec{e}_{i_1}, \vec{e}_{i_2} \cdots \vec{e}_{i_n})}\end{aligned}$$

If $\exists_{j,k}$ s.t. $\vec{e}_{i_j j} = \vec{e}_{i_k k}$, $D(\vec{e}_{i_1 1}, \cdots \vec{e}_{i_n n}) = 0$.

① $\longrightarrow$ $= \sum_{\sigma \in Perm(n)} a_{\sigma(1),1} a_{\sigma(2),1} \cdots a_{\sigma(n),n} \underbrace{D(\vec{e}_{\sigma(1)}, \vec{e}_{\sigma(2)}, \cdots \vec{e}_{\sigma(n)})}$

$D(\vec{e}_1, \cdots \vec{e}_n)$ after N interchanges.

② $\longrightarrow$ $= \sum_{\sigma \in Perm(n)} a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}\, sign(\sigma)$

$\rightarrow$ formal definition of Determinant.

MA54. 3.5. Fri. #2

① **Permutation** : rearrangement of $(1, 2, 3, \dots, n)$

math description : $\sigma : \{1, 2 \dots n\} \mapsto \{1, 2, \dots n\}$
s.t. σ assumes all values &
assumes each values once.

σ - bijections. $\text{Perm}(n)$

② **Def** if $\sigma \in \text{Perm}$, $\text{sign}(\sigma) = (-1)^N$
where N is # of interchanges to get from $\sigma(1), \sigma(2) \cdots \sigma(n)$
to $1, 2, \dots n$ (to get from $1, 2, \dots n$ to $\sigma(1), \sigma(2), \cdots \sigma(n)$)
Sign is well defined.

Def if $\sigma \in \text{Perm}$, $\text{signum}(\sigma) = (-1)^K$
K = number of pairs in "wrong order".
= # of (j, k) s.t. $j < k$ but $\sigma(j) > \sigma(k)$

a. If we change two neighbors, we change signum
b. Any interchange can be represented as an odd # of elementary interchanges

MA54 3.5. Fri #3

Two ways of representing Perm as functions.

1. $\sigma(1), \sigma(2), \cdots \sigma(n)$ - new order of $1, 2, \cdots n$

   e.g. $1, 2, 3, 4 \to 2, 3, 4, 1$

   $\sigma(1)=2,\ \sigma(2)=3,\ \sigma(4)=1,\ \sigma(3)=4$

2. $\tau(k)$ - new position of element # k.

   e.g. $1, 2, 3, 4 \to 2, 3, 4, 1$

   $\tau(1)=4,\ \tau(2)=1,\ \tau(3)=2,\ \tau(4)=3$

$\tau \circ \sigma$: Identity

$\tau = \sigma^{-1}$