Spectral theory

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$\boxed{A \ n\times n}$

Eigenvalues & eigenvectors

$$\begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix}^{2004}$$

$$\boxed{\begin{array}{l} \vec{x}_n = A\vec{x}_{n-1} \quad \vec{x}_0 \text{-given} \\ \vec{x}_n = A^n \vec{x}_0 \end{array}}$$

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} \qquad \vec{x}\,'(t) = A\vec{x}(t)$$

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Want to be able to find powers of matrices.

Suppose for some $\vec{x} \neq \vec{0}$

$$A\vec{x} = \lambda\vec{x}$$

$$A^2\vec{x} = A(\lambda\vec{x}) = \lambda^2\vec{x}$$

$$A^3\vec{x} = \lambda^3\vec{x}$$

$$\vdots$$

$$A^n\vec{x} = \lambda^n\vec{x}$$

$$\vdots$$

Note: $\vec{x} \neq \vec{0}$ essential

Def

λ is called an eigenvalue of A $(n\times n)$ if $\exists \vec{x} \neq \vec{0}$ st. $A\vec{x} = \lambda\vec{x}$

Vector $\vec{x}$ is called an eigenvector (corresponding to λ)

Def

Spectrum of A $(\sigma(A))$ is the set of all eigenvalues

How to find eigenvalues

$A\vec{x} = \lambda \vec{x}$

$A\vec{x} - \lambda \vec{x} = \vec{0}$

$(A - \lambda I)\vec{x} = \vec{0}$ λ is an eigenvalue iff this equation has a nontrivial solution.

$\iff A - \lambda I$ not invertible

$\iff \underbrace{\det(A - \lambda I)}_{p(\lambda)\text{ - polynomial of degree } n} = 0$ Characteristic equation

Characteristic polynomial

Eigenvalues - roots of characteristic polynomial

Ex. $\begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 8 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 16$

$(1-\lambda)^2 - 16 = 0$

$(1-\lambda)^2 = 16$

$1 - \lambda = \pm 4$

$\boxed{\lambda = 5,\ \lambda = -3}$

Finding the eigenvectors

$\lambda = 5$

$A - \lambda I = A - 5I = \begin{bmatrix} -4 & 2 \\ 8 & -4 \end{bmatrix}$ Now just find null space of $A - 5I$

$(A - 5I)\vec{x} = 0$ $\begin{bmatrix} -4 & 2 \\ 8 & -4 \end{bmatrix} \sim \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$

x_2 - free

$x_1 = \frac{1}{2}x_2$

$\vec{x} = x_2 \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

The eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

If you see that $\dim \ker(A - \lambda I) = 1$, you can just guess a solution.

$\lambda = -3$; $A - \lambda I = A + 3I = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix} \sim \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix}$

The dimension of the null space is 1, so we can immediately guess $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\begin{pmatrix}1\\2\end{pmatrix}$, $\begin{pmatrix}1\\-2\end{pmatrix}$ form a basis $\mathcal{B}$

$$[A]_{\mathcal{B}\mathcal{B}} = \begin{bmatrix}5 & 0\\0 & -3\end{bmatrix}$$

$$[A]_{\mathcal{S}\mathcal{S}} = [I]_{\mathcal{S}\mathcal{B}} \underset{(\mathcal{B}\mathcal{B})}{\begin{bmatrix}5 & 0\\0 & -3\end{bmatrix}} [I]_{\mathcal{B}\mathcal{S}}$$

$$= \begin{bmatrix}1 & 1\\2 & -2\end{bmatrix}\begin{bmatrix}5 & 0\\0 & -3\end{bmatrix}\begin{bmatrix}1 & 1\\2 & -2\end{bmatrix}^{-1}$$

$$\underset{[A^n]_{\mathcal{S}\mathcal{S}}}{A^n} = \begin{bmatrix}1 & 1\\2 & -2\end{bmatrix}\underbrace{\begin{bmatrix}5^n & 0\\0 & (-3)^n\end{bmatrix}}_{[A^n]_{\mathcal{B}\mathcal{B}}}\begin{bmatrix}1 & 1\\2 & -2\end{bmatrix}^{-1}$$

$$A = SDS^{-1}$$

$$A^2 = SDS^{-1}SDS^{-1} = SD^2S^{-1}$$

Complex eigenvalues

$$A = \begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix} \quad \det(A - \lambda I) = \begin{vmatrix}1-\lambda & 2\\-2 & 1-\lambda\end{vmatrix} = (1-\lambda)^2 + 4 = 0$$

$$(1-\lambda)^2 = -4$$

$$1-\lambda = \pm 2i$$

$$1 \mp 2i = \lambda \qquad \lambda = 1 \pm 2i$$

$\lambda = 1+2i \quad A - \lambda I = \begin{pmatrix}-2i & 2\\-2 & -2i\end{pmatrix} \sim \begin{pmatrix}-2i & 2\\0 & 0\end{pmatrix} \quad \begin{pmatrix}1\\i\end{pmatrix}$

$\lambda = 1-2i \quad A - \lambda I \ldots\ldots\ldots \begin{pmatrix}1\\-i\end{pmatrix}$

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} 1+2i & 0 \\ 0 & 1-2i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1}$$

We got complex matrices during diagonalization, even if we started with a real one.