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Larissa
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Diagonalization

Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ basis in V

Let $A\vec{v}_k = \lambda_k \vec{v}_k$

write $[A]_{\mathcal{B}\mathcal{B}}$

First column: $[A\vec{v}_1]_{\mathcal{B}} = [\lambda_1 \vec{v}_1]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\lambda \vec{v}_1 = \lambda_1 \vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 + \ldots + 0\vec{v}_n$

Reminder: what is $[\vec{v}]_{\mathcal{B}}$?

$\vec{v}$ admits unique representation $\vec{v} = \sum_{k=1}^{n} x_k \vec{v}_k$

$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Second column: $[A\vec{v}_2]_{\mathcal{B}} = [\lambda_2 \vec{v}_2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \lambda_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$$[A]_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Linear Independence:

Theorem: Let $A\vec{v}_k = \lambda_k \vec{v}_k$, $k = 1, 2, \ldots r$ and let $\lambda_k \neq \lambda_j$ if $j \neq k$

then $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_r$ - L. I

<u>Proof!</u> $r=1$ trivial

A system of 1 non-zero vector is L.I.

$r=2$ $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 = \vec{0}$ $(*)$

$$(A-\lambda_2 I)(\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2) = \vec{0}$$

$$\alpha_1\lambda_1\vec{v}_1 - \lambda_2\alpha_1\vec{v}_1 + \vec{0} = \vec{0}$$

$$\alpha_1\underbrace{(\lambda_1-\lambda_2)}_{\neq 0}\vec{v}_1 = \vec{0} \Rightarrow \alpha_1 = 0$$

$$(*)\ 0\vec{v}_1 + \alpha_2\vec{v}_2 = \vec{0} \Rightarrow \alpha_2 = 0$$

$$\Rightarrow \vec{v}_1\ \vec{v}_2 \text{ L.I.}$$

Suppose theorem is true for $r=1$

Let us prove it for r.

Let $\sum_{k=1}^{r}\alpha_k\vec{v}_k = \vec{0}$ $(**)$

$$(A-\lambda_r I)\left(\sum_{k=1}^{r}\alpha_k\vec{v}_k\right) = \vec{0}$$

$$\sum_{k=1}^{r-1}\alpha_k(\lambda_k\vec{v}_k - \lambda_r\vec{v}_k) = \vec{0}$$

$$\sum_{k=1}^{r-1}\alpha_k(\lambda_k-\lambda_r)\vec{v}_k = 0$$

We assumed $\vec{v}_1 \ldots \vec{v}_{r-1}$ is L.I.

$$\Rightarrow \alpha_k\underbrace{(\lambda_k-\lambda_r)}_{\neq 0} = 0 \quad k=1,2,\ldots r-1$$

$$\Rightarrow \alpha_k = 0 \quad k=1,2,\ldots r-1$$

$(**)$ $\alpha_r\vec{v}_r = \vec{0} \Rightarrow \alpha_r = 0 \Rightarrow$ All $\alpha_k = 0 \Rightarrow$ L.I.

Cor. Let $A : V \to V$, $\dim V = n$

If A has n distinct eigenvalues

$\Rightarrow$ A can be diagonalized.

Multiplicity of Eigenvalues

$p(z)$ - polynomial

λ - root $\quad p(\lambda) = 0$

$z - \lambda$ divides $p(z)$

$$p(z) = (z - \lambda)\, q(z)$$

It may happen λ is root of $q(z)$, $q(\lambda) = 0$.
Then $z - \lambda$ divides q and $(z-\lambda)^2$ divides p

Def: $p(z)$, λ-root, $p(\lambda) = 0$

mult of λ = maximal $k : (z-\lambda)^k$ divides $p(z)$

$$p(z) = \sum_{k=1}^{n} \alpha_k z^k \quad a_n \neq 0$$

$$p(z) = a_n (z - \lambda_1)^{k_1} (z - \lambda_2)^{k_2} \dots (z - \lambda_r)^{k_r}$$

$$k_1 + k_2 + \dots + k_r = n$$

roots (eigenvalues) counting multiplicities

roots	mult
1	2
2	2
3	1

1, 1, 2, 2, 3

Def: Mult. of eigenvalue - same as mult. of a root of characteristic polynomial $\Rightarrow$ Algebraic multiplicity

Geometric Multiplicity:
λ - e. value $\quad Ker(A - \lambda I)$ - eigen space
$\dim(Ker(A - \lambda I))$ - geom. mult.

Theorem: Let $A: V \to V$, λ_k - e. values
A is diagonalizable iff $\forall$ e. value λ_k
Alg. mult. = geom. mult.

Theorem:

1. Det A - product of e. values counting mult.
2. trace A - sum of e. values counting mult.

Remark: geom. mult. $\leq$ alg. mult.