Inner product spaces

$\mathbb{R}^3 \quad \vec{x}\cdot\vec{y} = x_1y_1 + x_2y_2 + x_3y_3$

length of $\vec{x} = \sqrt{x_1^2+x_2^2+x_3^2}$

$\vec{x} = \sqrt{x_1^2+x_2^2}$ in $\mathbb{R}^2$

<u>Norm</u> $\|\vec{x}\| = \sqrt{\vec{x}\cdot\vec{x}}$

$\mathbb{R}^n$ inner product

$$(\vec{x},\vec{y}) = \sum_{k=1}^{n} x_k y_k$$

Norm $\|\vec{x}\| = \sqrt{(\vec{x},\vec{x})} = \sqrt{\sum_{k=1}^{n} x_k^2}$

$\mathbb{C}^n \quad \vec{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \quad z_k = \mathbb{C} \qquad (z = x+yi) \quad (i^2=-1)$

$x = \mathrm{Re}\, z \qquad y = \mathrm{Im}\, z$

$$\frac{1+i}{1+2i} = \frac{(1+i)(1-2i)}{(1+2i)(1-2i)} = \frac{1-2i+i-i\cdot 2i}{1^2+2^2} = \frac{1+3-i}{5} = \frac{4-i}{5}$$

$z\bar{z} = x^2+y^2 = |z|^2 \qquad (\bar{z} = x-yi)$

$$\vec{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} x_1+iy_1 \\ x_2+iy_2 \\ \vdots \\ x_n+iy_n \end{pmatrix} \qquad \|z\| = \sqrt{\sum_{k=1}^{n}(x_k^2+y_k^2)} \text{ want}$$

Define $(\vec{z},\vec{w}) = \sum_{k=1}^{n} z_k \overline{w_k}$

$$(\vec{z},\vec{z}) = \sum_{k=1}^{n} z_k \bar{z}_k = \sum_{k=1}^{n}(x_k^2+y_k^2)$$

$$\|\vec{z}\| = \sqrt{(\vec{z},\vec{z})}$$

Properties of inner product:

1. $(\vec{v},\vec{y}) = \overline{(\vec{y},\vec{v})}$ Conjugate symmetry

$(\vec{x},\vec{v}) = (\vec{y},\vec{x})$ for $\mathbb{R}^n$

2. $(\alpha\vec{x}+\beta\vec{y},\vec{z}) = \alpha(\vec{x},\vec{z}) + \beta(y,\vec{z})$ Linearity

2'. $(\vec{v},\alpha\vec{y}+\beta\vec{z}) = \bar{\alpha}(\vec{v},\vec{y}) + \bar{\beta}(\vec{v},\vec{z})$

3. $(\vec{v},\vec{v}) \geq 0$ Nonnegativity

4. $(\vec{v},\vec{v}) = 0 \rightarrow \vec{v} = 0$

Def V suppose we have an inner product on V $\vec{x},\vec{y} \rightarrow (\vec{x},\vec{y})$ satisfying 1–4.

Inner product space

Ex V P or P_n

$(f,g) = \int_0^1 f(t)\overline{g(t)}\,dt$

Define a norm

$\|\vec{v}\| = \sqrt{(\vec{x},\vec{x})}$

1. $\|\vec{v}\| \geq 0$

2. $\|\alpha\vec{x}\| = |\alpha|\,\|\vec{v}\|$ $\quad \|\alpha\vec{x}\|^2 = (\alpha\vec{x},\alpha\vec{x}) = \alpha\bar{\alpha}(\vec{x},\vec{x})$

$= |\alpha|^2\|\vec{x}\|^2$

Homogeneity

3. $\|\vec{v}+\vec{y}\| \leq \|\vec{v}\| + \|\vec{y}\|$ Triangle inequality

4. $\|\vec{v}\| = 0 \Rightarrow \vec{v} = \vec{0}$