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Cauchy – Schwarz inequality
(Bunyakowsky)

$|(\vec{x},\vec{y})| \le \|\vec{x}\| \, \|\vec{y}\|$

• Knew for $\mathbb{R}^2$ or $\mathbb{R}^3$

$\vec{x} \cdot \vec{y} = \|\vec{x}\| \cdot \|\vec{y}\| \cdot \cos \alpha$

Pf: Real case

$$0 \le (\vec{x}+t\vec{y}, \vec{x}+t\vec{y}) = \|\vec{x}\|^2 + (t\vec{y},\vec{x}) + (\vec{x}, t\vec{y}) + \|t\vec{y}\|^2$$
$$= \|\vec{x}\|^2 + t(\vec{y},\vec{x}) + t(\vec{x},\vec{y}) + t^2\|\vec{y}\|^2$$
$$= \|\vec{x}\|^2 + 2t(\vec{x},\vec{y}) + t^2\|\vec{y}\|^2$$

$p = a + 2bt + ct^2$

$p' = 2b + 2ct = 0$

$t = -\frac{b}{c}$ $\rightarrow$ $t = \dfrac{-(\vec{x},\vec{y})}{\|\vec{y}\|^2}$

$$0 \le \|\vec{x}\|^2 - \frac{2(\vec{x},\vec{y})^2}{\|\vec{y}\|^2} + \frac{(\vec{x},\vec{y})^2}{(\|\vec{y}\|^2)^2}\|\vec{y}\|^2 = \|\vec{x}\|^2 - \frac{(\vec{x},\vec{y})^2}{\|\vec{y}\|^2}$$

$$0 \le \|\vec{x}\|^2\|\vec{y}\|^2 - (\vec{x},\vec{y})^2$$

$\therefore (\vec{x},\vec{y}) \le \|\vec{x}\|\|\vec{y}\|$

Complex Case

$$0 \le (\vec{x}+t\vec{y}, \vec{x}+t\vec{y}) = \|\vec{x}\|^2 + t(\vec{y},\vec{x}) + \bar{t}(\vec{x},\vec{y}) + |t|^2\|\vec{y}\|^2$$

• you could replace y by $\alpha\vec{y}$, $|\alpha| = 1$ to make $(\vec{x},\vec{y})$ real. But don't do that... just put $\rightarrow$ $t = \dfrac{-(\vec{x},\vec{y})}{\|\vec{y}\|^2}$

$$= \|\vec{x}\|^2 - \frac{(\vec{x},\vec{y})\overline{(\vec{x},\vec{y})}}{\|\vec{y}\|^2} - \frac{\overline{(\vec{x},\vec{y})}(\vec{x},\vec{y})}{\|\vec{y}\|^2} + \frac{|(\vec{x},\vec{y})|^2}{\|\vec{y}\|^4} \cdot \|y\|^2$$

$$= \|\vec{x}\|^2 - \frac{|(\vec{x},\vec{y})|^2}{\|\vec{y}\|^2} \ge 0$$

* only the complex case is needed for the formal proof.

$\therefore |(\vec{x},\vec{y})| \le \|\vec{x}\| \cdot \|\vec{y}\|$ $\square$

Proof of triangle inequality

$$\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 + \underbrace{(\vec{x},\vec{y}) + (\vec{y},\vec{x})}_{2\mathrm{Re}(\vec{x},\vec{y})}$$

$$\leq \|\vec{x}\|^2 + \|\vec{y}\|^2 + |(\vec{x},\vec{y})| + |(\vec{y},\vec{x})|$$

$$\leq \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\|\vec{x}\|\,\|\vec{y}\|$$

$$\leq (\|\vec{x}\| + \|\vec{y}\|)^2 \qquad \square$$

Orthogonality, orthogonal, & orthonormal bases

Def. $\vec{x}, \vec{y}$ are orthogonal if $(\vec{x},\vec{y}) = 0$ $\quad (\vec{x} \perp \vec{y})$

Pythagorean Theorem

If $\vec{x} \perp \vec{y}$ then $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$

Pf $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 + \underbrace{\cancel{2\mathrm{Re}(\vec{x},\vec{y})}}_{0}$

Generalized Pythagorean Theorem

$\vec{v}_1\, \vec{v}_2 \ldots \vec{v}_r$, $\vec{v}_j \perp \vec{v}_k \quad \forall j,k \quad k \neq j$

then $\left\|\sum_{k=1}^{r} \alpha_k \vec{v}_k\right\|^2 = \sum_{k=1}^{r} |\alpha_k|^2 \|\vec{v}_k\|^2$

orthogonal system: system of vectors in which all the vectors are orthogonal to each other.

↳ Corollary

Any orthogonal system of non-zero vectors is linearly independent.