Orthogonal basis.

$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$

$\vec{v}_j \perp \vec{v}_k$ if $k \neq j$

orthonormal basis

$(ONB) = OB + \|\vec{v}_k\| = 1$

$\vec{v}_1, \vec{v}_2, \dots \vec{v}_n$ - basis

$\vec{v} = \sum_{k=1}^{n} \alpha_k \vec{v}_k$.

If OB than $\alpha_k = \dfrac{(\vec{v}, \vec{v}_k)}{\|\vec{v}_k\|^2}$

$$\vec{v} = \sum_{k=1}^{n} \underbrace{\frac{(\vec{v}, \vec{v}_k)}{\|\vec{v}_k\|^2}}_{\text{a Number.}} \cdot \vec{v}_k$$

$(\vec{v}_j, \vec{v}_k) = \delta_{j,k} = \begin{Bmatrix} 0 & j \neq k \\ 0 & j = k \end{Bmatrix}$

Kroniker's delta.

ONB

$\alpha_k = (\vec{v}, \vec{v}_k)$

$\vec{v} = \sum_{k=1}^{n} (\vec{v}, \vec{v}_k) \cdot \vec{v}_k$

ex:

$\boxed{\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \|\vec{v}\| = \sqrt{3}}$

Orthogonal projection.

Def: E subspace of V

$\vec{v} \in V$

The orthogonal projection of $\vec{v}$ on to E is a vector $\vec{w}$ such that.

① $\vec{w} \in E$

② $\vec{v} - \vec{w} \in E$ $\qquad (\vec{w} = P_E \vec{v})$

Thm Let $\vec{w} = P_E \vec{v}$.

Than $\vec{w}$ minimizes the distance from $\vec{v}$ to E, i.e. $\forall \vec{x} = E$

$$\|\vec{v} - \vec{x}\| \geq \|\vec{v} - \vec{w}\| = \|\vec{v} - P_E \vec{v}\|$$

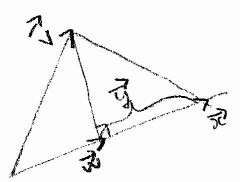
Moreover, equality occurs iff $\vec{x} = \vec{w}$.

Why do we care about $P_E \vec{v}$.

Corollary Orthog proj is Uniq.

Pf Take $\vec{x} \in E$

$\vec{y} = \vec{w} - x$



$\vec{v} - \vec{x} = \vec{v} - \vec{w} + \underbrace{\vec{w} + \vec{x}}$

$= \vec{v} - \vec{w} + \vec{y}$

$\vec{v} - \vec{w} \perp E$

$\Rightarrow \vec{v} - \vec{w} \perp \vec{y}$

$\Rightarrow \|\vec{v} - \vec{x}\|^2 = \|\vec{v} - \vec{w}\|^2 + \|\vec{y}\|^2 \geq \|\vec{v} - \vec{w}\|^2$

$=$ iff $\vec{y} = 0 \quad \vec{x} = \vec{w}$

How to find $P_E \vec{v}$.

Assume that $\vec{v}_1 \cdot \vec{v}_2, \ldots \vec{v}_r$ — OB in E.

Theorem $\vec{v}_1 \ldots \vec{v}_r$ — OB in E

$$\Rightarrow P_E \vec{v} = \sum_{k=1}^{r} \alpha_k \vec{v}_k$$

$$\alpha_k = \frac{(\vec{v}, \vec{v}_k)}{\|\vec{v}_k\|^2}$$

$$P_E \vec{v} = \sum_{k=1}^{r} \frac{(\vec{v}, \vec{v}_k)}{\|\vec{v}_k\|^2} \cdot \vec{v}_k$$

$\boxed{\text{L1 Let } \vec{v}_1 \ldots \vec{v}_r \text{ generating system for E. Then } \vec{v} \perp E \text{ iff } \vec{v} \perp \vec{v}_k \ \forall k = 1, 2, \ldots r}$

Pf of theorem. Need to show

1. $\vec{w} \in E$ — trivial
2. $\vec{v} - \vec{w} \perp E$ $\quad \vec{v} - \vec{w} \perp \vec{v}_j \ \forall j$

$$\left(\vec{v} - \sum_{k=1}^{r} \alpha_k \vec{v}_k, \vec{v}_j\right) = (v, \vec{v}_j) - \alpha(\vec{v}_j, \vec{v}_j)$$

$$= (\vec{v}, \vec{v}_j) - \frac{(\vec{v}, \vec{v}_j)}{\cancel{\|v_j\|^2}} \cancel{\|v_j\|^2} = 0$$

$$A^* = \overline{A^T}$$

$$(\vec{v}, \vec{w}) = \vec{w}^* \vec{v}$$

Example: $A = \begin{pmatrix} 1 & 1+2i \\ 1-3i & 1+i \end{pmatrix}$

$$A^* = \begin{pmatrix} 1 & 1+3i \\ 1-2i & 1-i \end{pmatrix}.$$

$$P_E = \sum_{k=1}^{r} \frac{\vec{v}_k \vec{v}_k^*}{\|v_k\|^2} = \sum \frac{1}{\|\vec{v}_k\|^2} \vec{v}_k \vec{v}_k^*$$

$$\vec{v}_k \vec{v}_k^* \vec{v} = \vec{v}_k (\vec{v}_k^* \vec{v}) = \vec{v}_k (\vec{v}, \vec{v}_k)$$